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The Distribution of Intervals between Zeros of a Stationary Random Function

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THE DISTRIBUTION OF INTERVALS BETWEEN ZEROS OF A STATIONARY RANDOM FUNCTION

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The probability density P_m of the spacing between the ith zero and the $(i+m+1)$th zero of a stationary, random function $f(t)$ (not necessarily Gaussian) is expressed as a series, of a type similar to that given by Rice (1945) but more rapidly convergent. The partial sums of the series provide upper and lower bounds successively for P_m. The series converges particularly rapidly for small spacings τ. It is shown that for fixed values of τ, the density $P_m(\tau)$ diminishes more rapidly than any negative power of m.

The results are applied to Gaussian processes; then the first two terms of the series for $P_m(\tau)$ may be expressed in terms of known functions. Special attention is paid to two cases:

(1) In the 'regular' case the covariance function $\psi(t)$ is expressible as a power series in t; then $P_m(\tau)$ is of order $\tau^{\frac{1}{2}(m+2)(m+3)-2}$ at the origin, and in particular $P(\tau)$ is of order τ (adjacent zeros have a strong mutual repulsion). The first two terms of the series give the value of $P_0(\tau)$ correct to τ^{18}.

(2) In a singular case, the covariance function $\psi(t)$ has a discontinuity in the third derivative. This happens whenever the frequency spectrum of $f(t)$ is O (frequency)$^{-4}$ at infinity. Then $P_m(\tau)$ is

shown to tend to a positive value $P_m(0)$ as $\tau \to 0$ (neighbouring zeros are less strongly repelled). Upper and lower bounds for $P_m(0)$ ($m = 0, 1, 2, 3$) are given, and it is shown that $P_0(0)$ is in the neighbourhood of $1{\cdot}155\psi'''/(-6\psi'')$. The conjecture of Favreau, Low & Pfeffer (1956) according to which in one case $P_0(\tau)$ is a negative exponential, is disproved.

In a final section, the accuracy of other approximations suggested by Rice (1945), McFadden (1958), Ehrenfeld *et al.* (1958) and the present author (1958) are compared and the results are illustrated by computations, the frequency spectrum of $f(t)$ being assumed to have certain ideal forms: a low-pass spectrum, band-pass spectrum, Butterworth spectrum, etc.

Introduction

The problem of finding the statistical distribution of the intervals τ between zeros of a stationary, random function $f(t)$ is one with many physical applications, for example, to noise in electrical circuits (Rice 1944, 1945; McFadden 1956, 1958), sea waves (Ehrenfeld *et al.* 1958), microseisms (Longuet-Higgins 1953) or irregularities in the ionosphere (Briggs & Page 1955; Longuet-Higgins 1957). Yet in one most useful case, when $f(t)$ itself is Gaussian, only approximate solutions to the problem have been found. One such solution was given in a previous paper (1958). In the present paper the solution is expressed in the form of a series, in which each term is an integral of the joint probability

$$W(+,-,-,\ldots,-)\,\mathrm{d}t_1\ldots\mathrm{d}t_n$$

that $f(t)$ have an up-crossing in the infinitesimal interval $(t_1, t_1+\mathrm{d}t_1)$ and a down-crossing in the remaining $(n-1)$ intervals $(t_i, t_i+\mathrm{d}t_i)$ $(i = 2, 3, \ldots, n)$. The series is somewhat similar to one given earlier by Rice (1945) but converges more rapidly. Moreover, successive partial sums of the series provide upper and lower bounds to the required distribution $P_0(\tau)$. The present series leads to a much more accurate estimate of the behaviour of P_0 near the origin $\tau = 0$ and to a systematic comparison of other approximations that have been previously proposed.

Following Rice (1945), the probability density of the interval between the ith and the $(i+m+1)$th zero of $f(t)$ is denoted by $P_m(\tau)$; and the probability of exactly n zeros occurring in the interval $(t, t+\tau)$ is denoted by $p(n,\tau)$. In § 1 of this paper, some relations between the W and the $P_m(\tau)$ are proved, and series are obtained for both $P_m(\tau)$ and $p(n,\tau)$ in terms of the W. One result is to show that $P_m(\tau)$ decreases with m more rapidly than any negative power of m. The relations are quite general, and no assumption is made as to the Gaussian character of $f(t)$. It is assumed only that $f(t)$ is statistically stationary; that $-f(t)$ is statistically equivalent to $f(t)$ (i.e. that $f(t)$ is statistically symmetric with regard to the axis $f = 0$); and that the various quantities defined actually exist.

In § 2 we obtain explicit expressions for the W in terms of the covariance function of $f(t)$, which is denoted by $\psi(t)$. By using some recent results of Kamat (1953) and Nabeya (1952) it is shown that $W(+,-,-)$ can in fact be expressed in terms of known functionals of $\psi(t)$, a fact not apparently used previously. Moreover, in special cases the W can be evaluated for any integral n.

The results are applied in § 3 to the case when $\psi(t)$ is itself a regular function at $t = 0$ (implying the differentiability of $f(t)$ up to all orders). Thus it is shown that for small intervals $|t_i - t_j|$

$$W(+,-,+,\ldots,(-)^{n-1}) \sim C_n \prod_{i<j}(t_j - t_i),$$

where C_n is a constant independent of the t_i. Hence the asymptotic behaviour of $P_m(\tau)$ and $p(n, \tau)$ for small τ is obtained. In particular it is shown that

$$P_m(\tau) \propto \tau^{\frac{1}{2}(m+2)(m+3)-2}$$

and
$$p(n, \tau) \propto \tau^{\frac{1}{2}n(n+1)}.$$

The power of τ increases rapidly with m or n, indicating a strong 'mutual repulsion' of neighbouring zeros of $f(t)$.

A very interesting singular case is studied in § 4, when the covariance function $\psi(t)$, instead of being regular at $t = 0$, has a finite discontinuity in the third derivative $f'''(t)$. This occurs, for example, whenever the spectral density of $f(t)$ is proportional to (frequency)$^{-4}$ at high frequencies, and some examples have been studied experimentally by Favreau, Low & Pfeffer (1956). In contrast to the regular case, it is shown that, for small spacings $|t_i - t_j|$,

$$W(+, -, +, \ldots, (-)^{n-1}) \sim \frac{F(t_1, t_2, \ldots, t_n)}{(t_3-t_1)(t_4-t_2)\ldots(t_n-t_{n-2})} \quad (n \geqslant 2),$$

where $F(t_1, t_2, \ldots, t_n)$ is a function of the $(t_i - t_j)$ lying between positive upper and lower bounds. It follows now that

$$P_m(\tau) \propto \tau^0 \quad (m \geqslant 0),$$

and
$$p(n, \tau) \propto \tau^2 \quad (n \geqslant 2),$$

so that, as $\tau \to 0$, $P_m(\tau)$ tends to a value $P_m(0)$ independent of τ.

Moreover, upper and lower bounds for $P_m(0)$ can be found. Thus

$$1{\cdot}147\alpha < P_0(0) < 1{\cdot}218\alpha,$$

where $\alpha = \psi'''(0+)/\{-6\psi''(0)\}$. This result enables us to disprove rigorously the 'exponential hypothesis' of Favreau *et al.* (1956), whereby the distribution of intervals τ for the function whose spectrum is $(1+\sigma^2)^{-2}$ is conjectured to be $\pi^{-1}\,e^{-\tau/\pi}$. For this would make

$$P_0(0) = 3\pi^{-1}\alpha = 0{\cdot}955\alpha,$$

in contradiction to the above inequality.

Some rough work shows that a close approximation to $P_0(\tau)$ is probably

$$P_0(0) \doteqdot 1{\cdot}155\alpha.$$

Lastly in § 5 we use the asymptotic expansions of $P_m(\tau)$, $p(n, \tau)$ and the W to compare the accuracy of the approximations proposed by Rice (1945), McFadden (1956, 1958), Ehrenfeld *et al.* (1958) and the present author (1958), especially in the neighbourhood of the origin. Numerical computation of the various approximations is also compared with experimental results of Blötekjaer (1958) and other authors.

1. General relations between $P_m(\tau)$, $p(n, \tau)$ and $W(S)$

Let $W(+, +, \ldots, +)\,dt_1 \ldots dt_n$ denote the probability that the function $f(t)$ have an up-crossing (zero-crossing with positive gradient) in each of the intervals

$$(t_i, t_i + dt_i) \quad (i = 1, \ldots, n);$$

by substituting a minus sign for any plus sign in $W(+, +, \ldots, +)$ we denote the corresponding probability for a down-crossing. Thus $W(+, -, +, \ldots, (-)^{n-1})\,dt_1 \ldots dt_n$ denotes the probability of alternate up-crossings and down-crossings in $(t_i, t_i + dt_i)$, the first being an up-crossing. It will be seen later that the W may in many cases be evaluated explicitly in terms of the covariance function of the random process $f(t)$.

In this section we shall derive some quite general relations between $P_m(\tau)$, $p(n,\tau)$ and the W, which are valid not only for Gaussian but for non-Gaussian processes.

1·1. *Generalization of a result of McFadden*

It was pointed out by McFadden (1958) that

$$\frac{W(+,-)}{W(+)} = P_0(\tau)+P_2(\tau)+P_4(\tau)+\ldots, \tag{1·1·1}$$

$$\frac{W(+,+)}{W(+)} = P_1(\tau)+P_3(\tau)+P_5(\tau)+\ldots, \tag{1·1·2}$$

where $\tau = (t_2-t_1)$. The proof is very simple: the left-hand side of (1·1·1), when multiplied by $\mathrm{d}t_2$, represents the probability that f has a down-crossing in $(t_2, t_2+\mathrm{d}t_2)$ given that it has an up-crossing at t_1. This down-crossing must be either the next zero of f, or the next but two, and so on. These mutually exclusive events correspond to the individual terms on the right of (1·1·1); hence the identity. A similar argument proves (1·1·2).

Corresponding to (1·1·1) and (1·1·2) we may establish the following three relations:

$$\int_{t_1<t_2<t_3} \frac{W(+,-,+)}{W(+)}\,\mathrm{d}t_2 = P_1(\tau)+2P_3(\tau)+3P_5(\tau)+\ldots, \tag{1·1·3}$$

$$\int_{t_1<t_2<t_3} \frac{W(+,-,-)}{W(+)}\,\mathrm{d}t_2 = P_2(\tau)+2P_4(\tau)+3P_6(\tau)+\ldots, \tag{1·1·4}$$

$$\int_{t_1<t_2<t_3} \frac{W(+,+,+)}{W(+)}\,\mathrm{d}t_2 = P_3(\tau)+2P_5(\tau)+3P_6(\tau)+\ldots, \tag{1·1·5}$$

where $\tau = (t_3-t_1)$. To prove (1·1·3), for example, consider

$$\frac{W(+,-,+)}{W(+)}\,\mathrm{d}t_2\,\mathrm{d}t_3.$$

This represents the probability that f has a down-crossing in $(t_2, t_2+\mathrm{d}t_2)$ and an up-crossing in $(t_3, t_3+\mathrm{d}t_3)$, given that it has an up-crossing at t_1. Now the up-crossing in $(t_3, t_3+\mathrm{d}t_3)$ is either the second zero after t_1 or the fourth or the sixth, etc. Suppose it is the $(2r)$th. Then the down-crossing in $(t_2, t_2+\mathrm{d}t_2)$ is either the first zero after t_1 or the third or the fifth, up to the $(2r-1)$th. But if the probabilities are each integrated with respect to t_2 from t_1 to t_3 each gives precisely $P_{2r-1}\,\mathrm{d}t_3$. The r possibilities together contribute $rP_{2r-1}\,\mathrm{d}t_3$. Hence the series (1·1·3); and similarly for (1·1·4) and (1·1·5).

We now prove the following general theorem. Let S denote any sequence of n signs, plus or minus, the first sign being $+$, and let s denote the number of times that the sequence changes sign. (For example, if $S = (+,-,+,-)$ then $s = 3$.) Then the expression $X(S)$ defined by

$$X(S) \equiv \int\ldots\int_{t_1<t_2<\ldots<t_n} \frac{W(S)}{W(+)}\,\mathrm{d}t_2\ldots\mathrm{d}t_{n-1} \tag{1·1·6}$$

is the sum of the series

$$X(S) = \sum_{r=0}^{\infty} \binom{n-2+r}{r} P_{2n-3-s+2r}(\tau), \tag{1·1·7}$$

where $\tau = (t_n-t_1)$ and $\binom{p}{q}$ denotes the coefficient of x^q in $(1+x)^p$.

Proof. The integrand in (1·1·6) represents the probability that the $(n-1)$ intervals $(t_i, t_i+\mathrm{d}t_i)$ $(i = 2, 3, \ldots, n)$, contain zero-crossings of $f(t)$ with gradients of the appropriate sign, given that f vanishes at t_1. Suppose that the last interval $(t_n, t_n+\mathrm{d}t_n)$ contains the $(k+1)$th zero of f after t_1. The remaining $(n-2)$ intervals $(t_2, t_2+\mathrm{d}t_2), \ldots (t_{n-1}, t_{n-1}+\mathrm{d}t_{n-1})$ must contain $(n-2)$ of the remaining k zeros between t_1 and t_n. Each distinct way of choosing these $(n-2)$ zeros contributes a term $P_k(\tau)$ to the integral $X(S)$. Hence

$$X(S) = \sum_k c_k P_k(\tau),$$

where c_k denotes the number of distinct ways of choosing the sequence S from a sequence S' of $(k+2)$ signs, alternately $+$ and $-$, so that the first and last signs of S^- correspond to those of S.

Now between each pair of successive signs of S that are both $+$ (or both $-$) there must be an odd number of signs of S. From the definition of s there are $(n-1-s)$ such cases, and so

$$k+2 = n+(n-1-s)+2r,$$

where $r = 0, 1, 2, \ldots$. The remaining r pairs of signs of S' may occur anywhere in the $(n-1)$ gaps between the signs of S. The number of distinct ways of disposing of these is

$$c_k = \binom{n-2+r}{r}.$$

On combining the last three equations we obtain (1·1·7).

It follows from the theorem that in the sequence S the only two relevant parameters are n and s. So we may write

$$X(S) = X_{n,s}. \tag{1·1·8}$$

The following special cases will be useful. When $S = (+, +, \ldots, +)$ then $s = 0$ and

$$X_{n,0} = \sum_{r=0}^{\infty} \binom{n-2+r}{r} P_{2n-3+2r}. \tag{1·1·9}$$

When $S = (+, -, -, \ldots, -)$ then $s = 1$ and

$$X_{n,1} = \sum_{r=0}^{\infty} \binom{n-2+r}{r} P_{2n-4+2r}. \tag{1·1·10}$$

When $S = (+, -, +, \ldots, (-)^{n-1})$ then $s = (n-1)$ and

$$X_{n,n-1} = \sum_{r=0}^{\infty} \binom{n-2+r}{r} P_{n-2+2r}. \tag{1·1·11}$$

From these series there follows also a useful result on the order of magnitude of $P_m(\tau)$ for large values of m (when τ is fixed). For the binomial coefficient in each case is

$$\binom{n-2+r}{r} = \frac{(n-2+r)(n-3+r)\ldots(1+r)}{(n-2)!} = O(r^{n-2})$$

as t tends to infinity. But if the series converges the individual terms must each tend to zero. So from (1·1·9), for example,

$$\lim_{r\to\infty} r^{n-2} P_{2n-3+2r} = 0.$$

In this expression the factor r^{n-2} may be replaced by $(2n-3+2r)^{n-2}$ without altering the limit. So

$$\lim_{m\to\infty} m^{n-2}P_m = 0 \tag{1·1·12}$$

if m tends to infinity through the odd values; and similarly for the even values.

Thus $P_m(\tau)$ tends to zero more rapidly than $m^{-(n-2)}$, τ being kept constant. Provided, then, that the $X_{n,s}$ exist for all values of n, it follows that $P_m(\tau)$ tends to zero more rapidly than any negative power of m.

1·2. *Series for $P_m(\tau)$*

Equations (1·1·1) to (1·1·5) may be written

$$\left.\begin{aligned} P_0 &= X(+,-)-(P_2+P_4+P_6+\dots),\\ P_1 &= X(+,+)-(P_3+P_5+P_7+\dots),\end{aligned}\right\} \tag{1·2·1}$$

and†

$$\left.\begin{aligned} P_1 &= X(+,-,+)-(2P_3+3P_5+4P_7+\dots),\\ P_2 &= X(+,-,-)-(2P_4+3P_6+4P_8+\dots),\\ P_3 &= X(+,+,+)-(2P_5+3P_7+4P_9+\dots).\end{aligned}\right\} \tag{1·2·2}$$

Rice (1945) and McFadden (1958) neglected $P_2, P_3, \dots$ and took as an approximation

$$\left.\begin{aligned} P_0 &\doteqdot X(+,-),\\ P_1 &\doteqdot X(+,+).\end{aligned}\right\} \tag{1·2·3}$$

However, by eliminating P_2 and P_3 from the right-hand side of (1·2·1) by means of equations (1·2·2) we have

$$\left.\begin{aligned} P_0 &= X(+,-)-X(+,-,-)+(P_4+2P_6+3P_8+\dots),\\ P_1 &= X(+,+)-X(+,+,+)+(P_5+2P_7+3P_9+\dots)\end{aligned}\right\} \tag{1·2·4}$$

so that a higher approximation, neglecting only $P_4, P_5, \dots$ is given by

$$\left.\begin{aligned} P_0 &\doteqdot X(+,-)-X(+,-,-),\\ P_1 &\doteqdot X(+,+)-X(+,+,+).\end{aligned}\right\} \tag{1·2·5}$$

It is easy to show that these approximations are the first in an infinite sequence. Equations (1·1·10), which involve only the even P_m, may be solved for P_0 by multiplying the first equation ($n = 2$) by 1, the next by -1, and so on up to $n = N$, and adding. On the right-hand side the coefficient of P_{2i}, when $i \leqslant N$, is

$$\binom{i}{0}-\binom{i}{1}+\binom{i}{2}-\dots+(-1)^i\binom{i}{i} = \begin{cases}1 & (i=0),\\ 0 & (0<i\leqslant N),\end{cases}$$

and when $i > N$ it is

$$\binom{i}{0}-\binom{i}{1}+\binom{i}{2}-\dots+(-1)^N\binom{i}{N} = (-1)^{N+1}\binom{i-1}{N}.$$

Hence $\quad X_{2,1}-X_{3,1}+X_{4,1}-\dots+(-)^N X_{N+2,1}$

$$= P_0+(-1)^N\left[P_{2N+2}+\binom{N+1}{1}P_{2N+4}+\binom{N+2}{2}P_{2N+6}+\dots\right]. \tag{1·2·6}$$

† The five equations (1·1·1) to (1·1·5), regarded as equations for the P_m, are not independent, for we have the identical relation

$$X(+,-,+)-X(+,+,+) = X(+,+).$$

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Thus provided the expression in square brackets tends to zero we have

$$P_0 = X_{2,1} - X_{3,1} + X_{4,1} - \dots, \tag{1·2·7}$$

and similarly

$$P_1 = X_{2,0} - X_{3,0} + X_{4,0} - \dots. \tag{1·2·8}$$

The approximations (1·2·3) and (1·2·5) correspond to the first two partial sums of these series.

Moreover, the remainder after $(N+1)$ terms of the series is

$$(-1)^{N+1}\left[P_{2N+2} + \binom{N+1}{1} P_{2N+4} + \binom{N+2}{2} P_{2N+6} + \dots\right] \tag{1·2·9}$$

which, since the P are all positive, has the same sign as $(-1)^{N+1}$. Hence the sums of the series (1·2·7) and (1·2·8) lie between any two successive partial sums.

The corresponding series for P_{2r} and P_{2r+1} $(r \geqslant 0)$ are found from (1·1·10) and (1·1·9) to be

$$\left.\begin{aligned} P_{2r} &= \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} X_{r+2+i,\,1}, \\ P_{2r+1} &= \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} X_{r+2+i,\,0} \end{aligned}\right\} \tag{1·2·10}$$

(the coefficients in the two series being identical). The solution of (1·1·11) for P_m is

$$P_m = (m+1) \sum_{i=0}^{\infty} (-1)^i \frac{(m+2i)!}{i!\,(m+i+1)!} X_{m+2+2i} \tag{1·2·11}$$

for all $m \geqslant 0$, where X_n is written more shortly for $X_{n,\,n-1}$. The solutions are valid provided each series is absolutely convergent.

1·3. *Relation to a series of Rice*

It is interesting to compare the series (1·2·7) for $P_0(\tau)$ with one stated by Rice (1945, equation (3·4–11)). His series may be written

$$P_0 = Y_2 - Y_3 + Y_4 - \dots, \tag{1·3·1}$$

where

$$Y_n \equiv \frac{1}{(n-2)!} \int_{t_1}^{t_n} \dots \int_{t_1}^{t_n} \frac{W(\pm, \pm, \dots, \pm)}{W(\pm)} \, dt_2 \dots dt_{n-1} \tag{1·3·2}$$

and $W(\pm, \pm, \dots, \pm)\,dt_1 \dots dt_n$ denotes the probability of a zero-crossing in each of the intervals $(t_i, t_i + dt_i)$ irrespective of the sign of $f'(t)$.

Now from the point of view of calculation Y_n is of a similar complexity to $X_{n,1}$, since each involves an $(n-2)$-fold integration of a probability density such as $W(S)$. On the other hand in the series (1·3·1) the first term neglected after N terms is Y_{N+3}, which is of order P_{N+1} (see below), whereas the remainder after N terms in the series (1·2·7) is only of order P_{2N+2}. Clearly then (1·2·7) is more rapidly convergent than (1·3·1).

The reason for this difference is apparently that in deriving (1·2·7) we have made use of the continuity of $f'(t)$ which implies that up-crossings and down-crossings follow one another alternately. In (1·3·1) no such property is used.

For completeness we now express Y_n in terms of the $P_m(\tau)$.

Since $-f(t)$ is assumed statistically equivalent to $f(t)$, the integrand in (1·3·2) may be replaced by $W(+, \pm, \pm, \ldots, \pm)/W(+)$; and further by the symmetry of the integrand with respect to $t_2, t_3, \ldots, t_{n-1}$ we have

$$Y_n = \int \ldots \int_{t_1<t_2<\ldots<t_n} \frac{W(+, \pm, \pm, \ldots, \pm)}{W(+)} \, dt_2 \ldots dt_{n-1}.$$

Now $W(+, \pm, \pm, \ldots, \pm)$ can be considered as the sum of 2^{n-1} expressions of the form $W(S)$, where S is a sequence of n signs such as was defined in § 1·1. Corresponding to any given value of s there are $\binom{n-1}{s}$ such sequences S. Hence

$$Y_n = \sum_{s=0}^{n-1} \binom{n-1}{s} X_{n,s}.$$

But $X_{n,s}$ or $X(S)$ is given by (1·1·7). Thus

$$Y_n = \sum_{s=0}^{n-1} \sum_{r=0}^{\infty} \binom{n-i}{s} \binom{n-2+r}{r} P_{n-2+s+2r}.$$

The coefficient of P_{n-2+i} may be summed by comparing coefficients of x^i in the expansion of the identity

$$(1+x)^{n-1}(1-x^2)^{-(n-1)} \equiv (1-x)^{-(n-1)}$$

in powers of x. Hence

$$Y_n = \sum_{i=0}^{\infty} \binom{n-2+i}{i} P_{n-2+i}. \tag{1·3·3}$$

The first term in this series is P_{n-2}, which proves our statement concerning the order of magnitude of Y_{N+3} made above.

The solution of equations (1·3·3) for the P_m is

$$P_m = \sum_{i=0}^{\infty} (-1)^i \binom{m+1}{i} Y_{m+2+i}, \tag{1·3·4}$$

of which (1·3·1) is the special case when $m = 0$.

1·4. *Series for $p(n, \tau)$*

McFadden (1958) has shown that $p(n, \tau)$ (the probability of exactly n zeros in the interval $(t, t+\tau)$) is related to $P_m(\tau)$ by the following set of equations:†

$$\left.\begin{aligned} p''(0,\tau) &= 2W(+)\,P_0, \\ p''(1,\tau) &= 2W(+)\,(P_1 - 2P_0), \\ p''(n,\tau) &= 2W(+)\,(P_n - 2P_{n-1} + P_{n-2}) \quad (n \geqslant 2), \end{aligned}\right\} \tag{1·4·1}$$

where a prime denotes differentiation with respect to τ. On substitution for P_n, P_{n-1}, P_{n-2} from equations (1·2·11) we have in the general case

$$p''(n,\tau) = 2W(+) \sum_{i=0}^{\infty} C_{n,i} X_{n+i}, \tag{1·4·2}$$

† The first of these relations is apparently due to Kohlenberg (1953).

where
$$C_{n,0} = 1$$
$$\left.\begin{aligned} C_{n,2r} &= (-1)^{r+1}(2r-n^2+n)\frac{(n-2+2r)!}{r!\,(n+r)!} \quad (r \geqslant 1), \\ C_{n,2r+1} &= (-1)^{r+1}\,2n\frac{(n-1+2r)!}{r!\,(n+r)!} \quad (r \geqslant 0). \end{aligned}\right\} \tag{1·4·3}$$

Now by definition
$$X_n = \int\!\!\ldots\!\!\int_{t_1<t_2<\ldots<t_n} \frac{W(+,-,+,\ldots,(-)^{n-1})}{W(+)}\,\mathrm{d}t_2\ldots\mathrm{d}t_n \tag{1·4·4}$$
which is a function of $\tau = (t_n - t_1)$. Hence
$$X_n(\tau) = \frac{1}{W(+)} I_n''(\tau), \tag{1·4·5}$$
where
$$I_n = \int\!\!\ldots\!\!\int_{0<t_1<\ldots<t_n<\tau} W(+,-,+,\ldots,(-)^{n-1})\,\mathrm{d}t_1\ldots\mathrm{d}t_n. \tag{1·4·6}$$
On substituting in (1·4·2) and integrating twice with respect to τ from $\tau = 0$ we have
$$p(n,\tau) = 2\sum_{i=0}^{\infty} C_{n,i} I_{n+i}(\tau) \tag{1·4·7}$$
provided the constants of integration vanish. The first term in this expansion is $2I_n$.

Let $R(t)$ denote the covariance function of the function $\xi(t)$ which equals 1 when $f(t) > 0$ and -1 when $f(t) < 0$. Rice (1944) showed that
$$R(t) = p(0,\tau) - p(1,\tau) + p(2,\tau) - \ldots.$$
By differentiating twice and using equations (1·4·1) one obtains
$$R''(\tau) = 8W(+)\,(P_0 - P_1 + P_2 - \ldots) \tag{1·4·8}$$
(McFadden 1958). From the first two equations of § 1·1 it follows that
$$R''(\tau) = 8[W(+,-) - W(+,+)]. \tag{1·4·9}$$

2. Evaluation of $W(S)$ for Gaussian processes

We now specialize the discussion to the case when $f(t)$ is Gaussian, and seek some explicit formulae for $W(S)$ in terms of the covariance function of $f(t)$.

2·1. *A general expression for* $W(S)$

Consider first the probability $W(+,+,\ldots,+)\,\mathrm{d}t_1\ldots\mathrm{d}t_n$ that $f(t)$ should have a zero up-crossing in each of the small intervals $(t_i, t_i+\mathrm{d}t_i)$ $(i = 1,\ldots,n)$. For convenience write
$$f(t_i) = \xi_i, \quad f'(t_i) = \eta_i \quad (i = 1,\ldots,n),$$
and let $p(\xi_1,\ldots,\xi_n;\eta_1,\ldots,\eta_n)$ denote the joint probability density of the ξ_i and η_i. Thus
$$p(\xi_1,\ldots,\xi_n;\eta_1,\ldots,\eta_n)\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n \tag{2·1·1}$$
is the probability that the ξ_i and η_i lie in given intervals $(\xi_i, \xi_i+\mathrm{d}\xi_i)$, $(\eta_i, \eta_i+\mathrm{d}\eta_i)$. Now if $f(t)$ has a zero-crossing in $(t_i, t_i+\mathrm{d}t_i)$, with gradient η_i, then $f(t_i)$ must lie in a small range of values of extent $|\eta_i|\,\mathrm{d}t_i$. Thus to obtain the probability $W(+,+,\ldots,+)\,\mathrm{d}t_1\ldots\mathrm{d}t_n$ we replace

$d\xi_i$ in (2·1·1) by $|\eta_i|\,dt_i$ and integrate over all positive values of the η_i. After dividing by $dt_1 \dots dt_n$ we have

$$W(+,+,\dots,+) = \int_0^\infty \dots \int_0^\infty |\eta_1 \dots \eta_n|\, p(0,\dots,0;\eta_1,\dots,\eta_n)\, d\eta_1 \dots d\eta_n. \tag{2·1·2}$$

For the covariance function of $f(t)$ we write

$$\overline{f(t)\, f(t+\tau)} = \psi(\tau).$$

The function $\psi(\tau)$ (or ψ_τ) is considered as given: it is the cosine transform of the spectral density of $f(t)$.

Then the covariance matrix of the $2n$ variables $\xi_1,\dots,\xi_n;\,\eta_1,\dots,\eta_n$ is

$$(\lambda_{ij}) = \begin{pmatrix} \psi_{11} & \dots & \psi_{1n} & \psi'_{11} & \dots & \psi'_{1n} \\ & & \vdots & \vdots & & \vdots \\ \psi_{n1} & \dots & \psi_{nn} & \psi'_{n1} & \dots & \psi'_{nn} \\ -\psi'_{11} & \dots & -\psi'_{1n} & -\psi''_{11} & \dots & -\psi''_{1n} \\ & & & \vdots & & \vdots \\ -\psi'_{n1} & \dots & -\psi'_{nn} & -\psi''_{n1} & \dots & -\psi''_{nn} \end{pmatrix}, \tag{2·1·3}$$

where $\psi_{ij} = \psi(t_i - t_j)$ and a prime denotes differentiation.

By the Gaussian hypothesis we have

$$p(\xi_1,\dots,\xi_n;\,\eta_1,\dots,\eta_n) = \frac{1}{(2\pi)^n\,\Delta^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2n} L_{ij}\xi_i\xi_j\right], \tag{2·1·4}$$

where $\xi_{n+i} = \eta_i$ and

$$\Delta = |(\lambda_{ij})|, \quad (L_{ij}) = (\lambda_{ij})^{-1}. \tag{2·1·5}$$

Substitution in (2·1·2) gives

$$W(+,+,\dots,+) = \frac{1}{(2\pi)^n\,\Delta^{\frac{1}{2}}} \int_0^\infty \dots \int_0^\infty |\eta_1 \dots \eta_n| \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} L_{n+i,\,n+j}\eta_i\eta_j\right] d\eta_1 \dots d\eta_n.$$

The summation in the last equation involves only the last n rows and columns of (L_{ij}). It is convenient to denote the inverse of this matrix by (μ_{ij})

$$(\mu_{ij}) = \begin{pmatrix} L_{n+1,\,n+1} & \dots & L_{n+1,\,2n} \\ \vdots & & \\ L_{2n,\,n+1} & \dots & L_{2n,\,2n} \end{pmatrix}^{-1}. \tag{2·1·7}$$

By Jacobi's theorem the (r,s)th element of this matrix is the bordered determinant

$$\mu_{r,s} = \begin{vmatrix} \psi_{11} & \dots & \psi_{1n} & \psi'_{1s} \\ \vdots & & \vdots & \vdots \\ \psi_{n1} & \dots & \psi_{nn} & \psi'_{ns} \\ -\psi'_{r1} & \dots & -\psi'_{rn} & -\psi''_{rs} \end{vmatrix} \div D, \tag{2·1·8}$$

where

$$D = \begin{vmatrix} \psi_{11} & \dots & \psi_{1n} \\ \vdots & & \vdots \\ \psi_{n1} & \dots & \psi_{nn} \end{vmatrix}. \tag{2·1·9}$$

The determinant of (μ_{ij}) is given by

$$|(\mu_{ij})| = \Delta/D. \tag{2·1·10}$$

(μ_{ij}) will be recognized as the covariance matrix of $(\eta_1, \ldots, \eta_n)$ given that

$$\xi_1 = \xi_2 = \ldots = \xi_n = 0.$$

For, if $p(\eta_1, \ldots, \eta_n | \xi_1, \ldots, \xi_n)$ denotes the conditional distribution of $(\eta_1, \ldots, \eta_n)$ for given values of $(\xi_1, \ldots, \xi_n)$ we have

$$p(\eta_1, \ldots, \eta_n | \xi_1, \ldots, \xi_n) = \frac{p(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)}{p(\xi_1, \ldots, \xi_n)},$$

where $p(\xi_1, \ldots, \xi_n)$ is the distribution of $(\xi_1, \ldots, \xi_n)$ only

$$p(\xi_1, \ldots, \xi_n) = \frac{1}{(2\pi)^{\frac{1}{2}n} D^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\xi_i\xi_j\right],$$

where (M_{ij}) is the inverse of (ψ_{ij}). Hence when the ξ_i vanish we have, using (2·1·10),

$$p(\eta_1, \ldots, \eta_n | 0, \ldots, 0) = \frac{1}{(2\pi)^{\frac{1}{2}n} |(\mu_{ij})|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} L_{n+i,n+j}\eta_i\eta_j\right] = Z(\boldsymbol{\eta}, \boldsymbol{\mu}),$$

say. With this notation (2·1·6) may be written

$$W(+, +, \ldots, +) = \frac{1}{(2\pi)^{\frac{1}{2}n} D^{\frac{1}{2}}} \int_0^\infty \ldots \int_0^\infty |\eta_1 \ldots \eta_n| \, Z(\boldsymbol{\eta}, \boldsymbol{\mu}) \, d\eta_1 \ldots d\eta_n. \tag{2·1·11}$$

It is convenient to introduce the normalized convariance matrix (ν_{ij}) whose (i, j)th element is

$$\nu_{ij} = \frac{\mu_{ij}}{(\mu_{ii}\mu_{jj})^{\frac{1}{2}}}. \tag{2·1·12}$$

Then on writing $$\zeta_i = (\mu_{ii})^{-\frac{1}{2}}\eta_i$$

in equation (2·1·11) so that (ν_{ij}) is the covariance matrix of the new variables ζ_i, we have

$$W(+, +, \ldots, +) = \frac{(\mu_{11}\mu_{22} \ldots \mu_{nn})^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n} D^{\frac{1}{2}}} J_n, \tag{2·1·13}$$

where $$J_n = \int_0^\infty \ldots \int_0^\infty \zeta_1 \ldots \zeta_n Z(\boldsymbol{\zeta}, \boldsymbol{\nu}) \, d\zeta_1 \ldots d\zeta_n. \tag{2·1·14}$$

Now $Z(\boldsymbol{\zeta}, \boldsymbol{\nu})$ is the ordinary normal probability density in the n variables ζ_i, with covariance matrix (ν_{ij}). Since the diagonal elements have been normalized (by equation (2·1·12)), J_n is a function only of the off-diagonal element ν_{ij} $(i \neq j)$.

Suppose that one of the zeros in the sequences (say the kth zero) is to be a down-crossing instead of an up-crossing. Then in calculating the corresponding probability density W we need only to take the range of integration of η_k in (2·1·2) from $-\infty$ to 0 instead of 0 to ∞. Equivalently, we may simply reverse the sign of the $(n+k)$th row and column of L_{ij}, and hence the kth row and column of (μ_{ij}) and of (ν_{ij}).

Hence to find $W(+, -, +, \ldots, (-)^{n-1})$, in which each alternate zero-crossing is a down-crossing, we have to multiply $L_{n+i,n+j}$ by $(-1)^{i+j}$ and hence also multiply μ_{ij} and ν_{ij} by $(-1)^{i+j}$.

2·2. *The cases* $n = 1$, 2 *and* 3

The case $n = 1$ is trivial, for then $Z(\boldsymbol{\zeta}, \boldsymbol{\nu})$ is the normal distribution for a single variate and

$$J_1 = \int_0^\infty \zeta \frac{e^{-\frac{1}{2}\zeta^2}}{(2\pi)^{\frac{1}{2}}} d\zeta = \frac{1}{(2\pi)^{\frac{1}{2}}}.$$

Since in (2·1·8) ψ'_{11} vanishes we have $\mu_{11} = -\psi''_{11} = -\psi''_0$ and so from (2·1·13)

$$W(+) = \left(\frac{\mu_{11}}{2\pi D}\right)^{\frac{1}{2}} J_1 = \frac{1}{2\pi}\left(\frac{-\psi''_0}{\psi_0}\right)^{\frac{1}{2}} \tag{2·2·1}$$

as is well known (Kac 1943).

Also well known is the case $n = 2$, when

$$J_2 = \frac{1}{2\pi}[(1-\nu_{12}^2)^{\frac{1}{2}} + \nu_{12}\cos^{-1}(-\nu_{12})]$$

(the angle being chosen so as to lie between 0 and π). This gives

$$W(+,+) = \frac{(\mu_{11}\mu_{22})^{\frac{1}{2}}}{4\pi^2(\psi_0^2-\psi_{12}^2)^{\frac{1}{2}}}[(1-\nu_{12}^2)^{\frac{1}{2}} + \nu_{12}\cos^{-1}(-\nu_{12})] \tag{2·2·2}$$

(Rice 1945, § 3·10). By changing the sign of ν_{12} we have

$$W(+,-) = \frac{(\mu_{11}\mu_{22})^{\frac{1}{2}}}{4\pi^2(\psi_0^2-\psi_{12}^2)^{\frac{1}{2}}}[(1-\nu_{12}^2)^{\frac{1}{2}} - \nu_{12}\cos^{-1}(\nu_{12})]. \tag{2·2·3}$$

Not so well known is the case $n = 3$. However, J_3 may be derived from some integrals calculated quite recently by Nabeya (1952) and Kamat (1953). One obtains

$$J_3 = \frac{1}{(2\pi)^{\frac{3}{2}}}[|(\nu_{ij})|^{\frac{1}{2}} + (s_1\alpha_1 + s_2\alpha_2 + s_3\alpha_3)],$$

where
$$s_1 = \cos^{-1}\frac{\nu_{31}\nu_{12}-\nu_{23}}{(1-\nu_{31}^2)^{\frac{1}{2}}(1-\nu_{23}^2)^{\frac{1}{2}}}, \qquad \alpha_1 = \nu_{31}\nu_{12}+\nu_{23}.$$

(s_2, s_3, etc., are obtained by cyclic permutation of the ν_{ij}). These angles are also to be taken in the range $(0, \pi)$. So from (2·1·13) we find

$$\left.\begin{aligned}
W(+,+,+) &= \frac{(\mu_{11}\mu_{22}\mu_{33})^{\frac{1}{2}}}{8\pi^3 D^{\frac{1}{2}}}[|(\nu_{ij})|^{\frac{1}{2}} + (s_1\alpha_1+s_2\alpha_2+s_3\alpha_3)],\\
W(+,-,+) &= \frac{(\mu_{11}\mu_{22}\mu_{33})^{\frac{1}{2}}}{8\pi^3 D^{\frac{1}{2}}}[|(\nu_{ij})|^{\frac{1}{2}} + (s_1-\pi)\alpha_1+s_2\alpha_2+(s_3-\pi)\alpha_3],\\
W(+,-,-) &= \frac{(\mu_{11}\mu_{22}\mu_{33})^{\frac{1}{2}}}{8\pi^3 D^{\frac{1}{2}}}[|(\nu_{ij})|^{\frac{1}{2}} + s_1\alpha_1+(s_2-\pi)\alpha_2+(s_3-\pi)\alpha_3].
\end{aligned}\right\} \tag{2·2·4}$$

2·3. *General values of n*

When $n > 3$, the integral J_n cannot be expressed in terms of known functions, in general. However, two particular cases in which this is possible may be stated here for later use.

First, if the covariances ν_{ij} all vanish when $i \neq j$, then $\zeta_1, \ldots, \zeta_n$ are statistically independent variates and

$$J_n = \left[\int_0^\infty \zeta\frac{e^{-\frac{1}{2}\zeta^2}}{(2\pi)^{\frac{1}{2}}}d\zeta\right]^n = \frac{1}{(2\pi)^{\frac{1}{2}n}}$$

giving
$$W = \frac{1}{(2\pi)^n}\frac{(\mu_{11}\ldots\mu_{nn})^{\frac{1}{2}}}{D^{\frac{1}{2}}}. \tag{2·3·1}$$

Secondly, if all the covariances ν_{ij} are unity, then $\zeta_1, \ldots, \zeta_n$ all reduce to the same Gaussian variate, giving

$$J_n = \int_0^\infty \zeta^n\frac{e^{-\frac{1}{2}\zeta^2}}{(2\pi)^{\frac{1}{2}}}d\zeta = \frac{1}{(2\pi)^{\frac{1}{2}}}2^{\frac{1}{2}(n-1)}\left(\frac{n-1}{2}\right)!$$

Hence
$$W = \frac{1}{2\pi^{\frac{1}{2}(n+1)}}\left(\frac{n-1}{2}\right)!\frac{(\mu_{11}\ldots\mu_{nn})^{\frac{1}{2}}}{D^{\frac{1}{2}}}. \tag{2·3·2}$$

3. Asymptotic expansions near the origin: ψ regular

In this section we evaluate the probabilities defined earlier, for small time intervals τ. It will be assumed that the covariance function $\psi(t)$ is regular at the origin

$$\psi(t) = \psi_0 + \frac{\psi''_0}{2!}t^2 + \frac{\psi^{\text{iv}}_0}{4!}t^4 + \ldots$$

(coefficients of the odd powers vanish, since $\psi(t)$ is an even function of (t).

3·1. *Expansion of $W(S)$*

Our first object is to obtain a multiple power series in the t_i for the probability density $W(S)$. Since W depends only on the covariance matrix (λ_{ij}) of equation (2·1·3), it is evidently a function of the time differences $(t_i - t_j)$. We shall see that the leading terms in W are homogeneous and of degree $\frac{1}{2}n(n-1)$ in the $(t_i - t_j)$.

We use the following lemma: if $F(x)$ is any function of x regular at $x = 0$, then the leading term in the expansion of

$$\begin{vmatrix} F(x_1+y_1) & \ldots & F(x_1+y_n) \\ \vdots & & \vdots \\ F(x_n+y_1) & \ldots & F(x_n+y_n) \end{vmatrix}$$

in powers of the x_i and y_i is

$$\begin{vmatrix} F(0) & F'(0) & \ldots & F^{(n-1)}(0) \\ F'(0) & F''(0) & \ldots & F^{(n)}(0) \\ \vdots & \vdots & & \vdots \\ F^{(n-1)}(0) & F^{(n)}(0) & \ldots & F^{(2n-2)}(0) \end{vmatrix} \frac{\prod_{i<j}(x_j - x_i)(y_j - y_i)}{[1!\,2!\ldots(n-1)!]^2}.$$

To prove this, first express each term as a Taylor series in the x_i

$$F(x_i + y_j) = F(y_j) + x_i F'(y_j) + \frac{x_i^2}{2!}F''(y_j) + \ldots.$$

Subtract the first row of the determinant from the remaining rows, taking out the factors $(x_2 - x_1), \ldots (x_n - x_1)$; then subtract the second row from the rows beneath it, taking out the factors $(x_3 - x_2), \ldots, (x_n - x_2)$; and so on. In the result write

$$F^{(i)}(y_j) = F^{(i)}(0) + y_j F^{(i+1)}(0) + \frac{y_j^2}{2!}F^{(i+2)}(0) + \ldots$$

and proceed similarly with the columns. This gives the result.

Setting $F = \psi$, $x_i = t_i$ and $y_i = -t_i$ in the lemma we find, from (2·1·9),

$$D \sim D_n \frac{\prod_{i<j}(t_j - t_i)^2}{[1!\,2!\ldots(n-1)!]^2}, \tag{3·1·1}$$

where

$$D_m = (-1)^{\frac{1}{2}m(m-1)} \begin{vmatrix} \psi_0 & \psi'_0 & \ldots & \psi_0^{(m-1)} \\ \psi'_0 & \psi''_0 & \ldots & \psi_0^{(m)} \\ \vdots & \vdots & & \vdots \\ \psi_0^{(m-1)} & \psi_0^{(m)} & \ldots & \psi_0^{(2m-2)} \end{vmatrix}. \tag{3·1·2}$$

It will be noticed that since the odd derivatives of ψ all vanish, every alternate element of D_m is zero, so that D_m may be factorized into two determinants

$$D_m = (-1)^{\frac{1}{2}m(m-1)} \begin{vmatrix} \psi_0 & \psi_0'' & \dots \\ \psi_0'' & \psi_0^{(4)} & \dots \\ \vdots & \vdots & \end{vmatrix} \times \begin{vmatrix} \psi_0'' & \psi_0^{\text{iv}} & \dots \\ \psi_0^{\text{iv}} & \psi_0^{(6)} & \dots \\ \vdots & \vdots & \end{vmatrix}.$$

By a similar method we may evaluate the leading term in μ_{rs} (equation (2·1·8)). One finds in fact

$$\mu_{rs} \sim \frac{D_{n+1}}{D_n} \frac{\prod_{i\neq r}(t_r-t_i)\prod_{j\neq s}(t_s-t_j)}{(n!)^2}. \tag{3·1·3}$$

Hence
$$\nu_{rs} = \frac{\mu_{rs}}{(\mu_{rr}\,\mu_{ss})^{\frac{1}{2}}} \sim \pm 1$$

according as μ_{rs} is positive or negative. Now since (ψ_{ij}) is a positive-definite matrix its determinant D must be positive, so that by (3·1·1) D_m is positive also. On the other hand when $t_1 < t_2 < \dots < t_n$ the product $\prod_{i\neq r}(t_r-t_i)$ has the same sign as $(-1)^{r+s}$, and so μ_{rs} has the same sign as $(-1)^{r+s}$. It follows that

$$\nu_{rs} \sim (-1)^{r+s}.$$

Now to calculate $W(+,-,+,\dots,(-)^{n-1})$ we recall that ν_{rs} was to be multiplied by $(-1)^{r+s}$. The elements of the corresponding covariance matrix thus all become equal to unity, in the limit, and so (2·3·2) applies. On substituting for μ_{ii} and D we find

$$W(+,-,+,\dots,(-)^{n-1}) \sim C_n \prod_{i<j}(t_j-t_i), \tag{3·1·4}$$

where
$$C_n = \frac{1!\,2!\dots(n-1)!}{2\pi^{\frac{1}{2}(n+1)}(n!)^n}\left(\frac{n-1}{2}\right)!\left(\frac{D_{n+1}^n}{D_n^{n+1}}\right)^{\frac{1}{2}}. \tag{3·1·5}$$

In particular when $n = 1$ and 2 we have the known results

$$W(+) = \frac{1}{2\pi}\frac{D_2^{\frac{1}{2}}}{D_1} \tag{3·1·6}$$

and
$$W(+,-) \sim \frac{1}{16\pi}\frac{D_3}{D_2^{\frac{3}{2}}}(t_2-t_1) \tag{3·1·7}$$

(cf. Rice 1945, § 3·4).

3·2. *Evaluation of $P_m(\tau)$*

The integral I_n defined by (1·4·6) can now be evaluated. We use the identity

$$\int\dots\int_{0<t_1<t_2<\dots<t_n<\tau} \prod_{i<j}(t_j-t_i)\,\mathrm{d}t_1\dots\mathrm{d}t_n = \frac{[1!\,2!\,3!\dots(n-1)!]^2}{1!\,3!\,5!\dots(2n-1)!}\tau^{\frac{1}{2}n(n+1)}, \tag{3·2·1}$$

a proof of which is given, for example, by Mehta (1960). From (1·4·6) and (3·1·4) we have then

$$I_n \sim \frac{[1!\,2!\,3!\dots(n-1)!]^2}{1!\,3!\,5!\dots(2n-1)!} C_n \tau^{\frac{1}{2}n(n+1)}, \tag{3·2·2}$$

which is of order $\tau^{\frac{1}{2}n(n+1)}$. From this it follows that

$$X_n = \frac{1}{W(+)} I_n'' = O(\tau^{\frac{1}{2}n(n+1)-2}). \tag{3·2·3}$$

Since the power of τ increases with n, one sees that $P_m(\tau)$ is given asymptotically by the first term in the series (1·2·11), i.e.

$$P_m(\tau) \sim X_{m+2}. \tag{3·2·4}$$

From the last three equations and (3·1·5) we have

$$P_m(\tau) \sim \frac{[1!\,2!\,3!\ldots(m+1)!]^2}{1!\,3!\,5!\ldots(2m+3)!}\,\frac{C_{m+2}}{C_1}\,\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\tau^{\frac{1}{2}(m+2)(m+3)}. \tag{3·2·5}$$

In particular

$$\left.\begin{aligned} P_0(\tau) &\sim \frac{C_2}{C_1}\tau = \frac{1}{8}\frac{D_1^{\frac{1}{2}}D_3}{D_2^{\frac{3}{2}}}\tau, \\ P_1(\tau) &\sim \frac{C_3}{C_1}\frac{\tau^4}{6} = \frac{1}{648\pi}\frac{D_1^{\frac{1}{2}}D_4^{\frac{3}{2}}}{D_2^{\frac{1}{2}}D_3^{2}}\tau^4, \end{aligned}\right\} \tag{3·2·6}$$

in agreement with Rice (1945) and Palmer (1956), respectively. In general we have

$$P_m(\tau) = O(\tau^{\frac{1}{2}(m+2)(m+3)-2}), \tag{3·2·7}$$

a power of τ that increases very rapidly with m.

3·3. *Evaluation of* $p(n, \tau)$

When $n = 0$ and 1 we have trivially

$$p(0, \tau) \sim 1 \tag{3·3·1}$$

and

$$p(1, \tau) \sim 2W(+)\,\tau = \frac{1}{\pi}\frac{D_2^{\frac{1}{2}}}{D_1}\tau \tag{3·3·2}$$

from (3·1·6).

When $n \geqslant 2$, both I_n and I'_n vanish at $\tau = 0$, and therefore (1·4·5) is valid provided both $p(n, 0)$ and $p'(n, 0)$ are zero. Both conditions are satisfied if we assume $p(n, \tau) = O(\tau^{1+\epsilon})$, where $\epsilon > 0$. In the series (1·4·7) the terms $I_{n+i}(\tau)$ are proportional to increasing powers of τ and hence $p(n, \tau)$ is given asymptotically by the first term

$$p(n, \tau) \sim 2I_n, \tag{3·3·3}$$

or on substitution from (3·2·2)

$$p(n, \tau) \sim 2\,\frac{[1!\,2!\,3!\ldots(n-1)!]^2}{1!\,3!\,5!\ldots(2n-1)!}\,C_n\,\tau^{\frac{1}{2}n(n+1)} \quad (n \geqslant 2). \tag{3·3·4}$$

For example

$$\left.\begin{aligned} p(2, \tau) &\sim \tfrac{1}{3}C_2\tau^3 = \frac{1}{48\pi}\frac{D_3}{D_2^{\frac{3}{2}}}\tau^3, \\ p(3, \tau) &\sim \tfrac{1}{90}C_3\tau^6 = \frac{1}{19\,440\pi^2}\frac{D_4^{\frac{3}{2}}}{D_3^{2}}\tau. \end{aligned}\right\} \tag{3·3·5}$$

Since $p(2, \tau)$ is of order τ^3 and not τ^2 we see that neighbouring zeros of $f(t)$ are not independent of one another. The effect may be called a 'mutual repulsion' of the zeros. It is connected with the property, seen in the previous section, that $P_0(\tau) \to 0$ as $\tau \to 0$, that is to say, small intervals τ are unlikely.

Moreover, as n increases, so the power of τ in $p(n, \tau)$ increases very rapidly.

A heuristic argument for the rapidly increasing power of τ may be given as follows. If $f(t)$ is to have n zeros in $(t, t+\tau)$ then by Rolle's theorem $f'(t)$ must have at least $(n-1)$ zeros in the same interval, and further $f''(t)$ must have at least $(n-2)$ zeros, and so on, till finally $f^{(n-1)}(t)$ must have at least one zero in the interval. Therefore (assuming the existence of $f^{(n)}$) $f^{(n-1)}$ must be of order τ throughout the interval, and by integration $f^{(n-2)}, f^{(n-3)}, \ldots, f$ must be of order $\tau^2, \tau^3, \ldots, \tau^n$, respectively. That is to say, $f^{(n-1)}, f^{(n-2)}, \ldots, f$ at some fixed point in the interval, must lie within ranges $\delta f^{(n-1)}, \delta f^{(n-2)}, \ldots, \delta f$ of order $\tau, \tau^2, \ldots, \tau^n$, respectively. The probability of such an event is of order

$$\delta f^{(n-1)}\,\delta f^{(n-2)} \ldots \delta f = O(\tau^{1+2+\ldots+n}) = O(\tau^{\frac{1}{2}n(n+1)}).$$

4. Asymptotic expansions near the origin: a singular case

We shall now seek expansions at the origin in a very interesting singular case. Instead of the Taylor series for $\psi(t)$ (equation (3·1)), suppose now that $\psi(t)$ has an expansion of the form

$$\psi(t) = \psi_0 + \frac{\psi_0''}{2!}t^2 + \frac{\psi_{0+}'''}{3!}|t|^3 + \ldots. \tag{4·1}$$

In other words, the third derivative of $\psi(t)$ possesses a finite discontinuity at the origin.† Some examples of such functions were studied experimentally by Favreau *et al.* (1956), and McFadden (1958) has considered them theoretically. They occur whenever the spectral density of $f(t)$ is of order (frequency)$^{-4}$ at high frequencies.

4·1. *Expansion of W(S)*

If the procedure of § 3·1 is attempted it is found that altogether fewer factors can be extracted from the determinants. For example, to evaluate D, defined by (2·1·9), we begin by subtracting row $(n-1)$ from row n of the determinant, then row $(n-2)$ from row $(n-1)$, and so on, in turn extracting the factors $(t_n - t_{n-1}), (t_{n-1} - t_{n-2}), \ldots, (t_2 - t_1)$; and similarly for the columns. The process is then repeated as far as row 2 only, and without extracting any factors. The leading term in the determinant is then seen to be

$$D \sim -\psi_0 \psi_0'' c^{n-2} (\tau_1 \tau_2 \ldots \tau_{n-1})^2 \, |\mathbf{A}| \quad (n > 2), \tag{4·1·1}$$

where we have written $\quad \frac{1}{6}\psi_{0+}''' = c, \qquad t_{i+1} - t_i = \tau_i, \qquad$ (4·1·2)

and where $\mathbf{A}$ is the $(n-2) \times (n-2)$ square matrix

$$\mathbf{A} = \begin{pmatrix} 4(\tau_1+\tau_2) & 2\tau_2 & 0 & \ldots & 0 \\ 2\tau_2 & 4(\tau_2+\tau_3) & 2\tau_3 & \ldots & 0 \\ 0 & 2\tau_3 & 4(\tau_3+\tau_4) & \ldots & 0 \\ \vdots & \vdots \quad \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 4(\tau_{n-2}+\tau_{n-1}) \end{pmatrix}. \tag{4·1·3}$$

When $n = 2$, $|\mathbf{A}|$ is replaced by unity in equation (4·1·1).

Instead of calculating (μ_{rs}) directly, it is rather more convenient to determine first $(L_{n+i,\,n+j})$. Now (L_{ij}) is the reciprocal of the covariance matrix (λ_{ij}), given by (2·1·3). The determinant of (λ_{ij}) is found, by a process similar to the above, to be

$$\Delta \sim -\psi_0 \psi_0'' c^{2n-2} (\tau_1 \tau_2 \ldots \tau_{n-1})^2 \, |\mathbf{E}|, \tag{4·1·4}$$

where $\mathbf{E}$ denotes the $(2n-2) \times (2n-2)$ square matrix made up as follows:

$$\mathbf{E} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^* & \mathbf{B} \end{pmatrix}$$

† The existence of $\psi''(t)$ is sufficient to ensure the joint distribution of $f(t)$ and $f'(t)$ as in § 2. If the expansion of $\psi(t)$ contains a term in $|t|$ then the mean density of zeros no longer exists in the usual sense. Such a case was considered by Siegert (1951). See also Rice (1958, § 9).

in which **A** is given by (4·1·3), **B** is the $n \times n$ matrix

$$\mathbf{B} = \left(\begin{array}{c|ccccc} 4\tau_1 & -2\tau_1 & 0 & 0 & \dots & 0 \\ \hline -2\tau_1 & 4\tau_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 4\tau_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 4\tau_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4\tau_{n-1} \end{array}\right)$$

and C is the $(n-2) \times n$ matrix

$$\mathbf{C} = \left(\begin{array}{c|ccccccc} -2\tau_1 & 4\tau_1 & 2\tau_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 4\tau_2 & 2\tau_3 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\tau_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4\tau_{n-3} & 2\tau_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 4\tau_{n-2} & 2\tau_{n-1} \end{array}\right)$$

($\mathbf{C}^*$ denotes the transpose of C). By subtracting the $(n-1+i)$th row of **E** from the ith row ($i = 1, 2, \dots, (n-2)$), and similarly for the columns we find

$$|\mathbf{E}| = 12^{n-1}(\tau_1\tau_2\dots\tau_{n-1})^2$$

and so
$$\Delta \sim 12^{n-1}(-\psi_0\psi_0'')\, c^{2n-2}(\tau_1\tau_2\dots\tau_{n-1})^4. \tag{4·1·5}$$

The quantities $\Delta L_{n+i\ n+j}$ involve the cofactors of the last n rows and columns of (λ_{ij}) and hence of **E**. Hence we find

$$(L_{n+i,\,n+j}) \sim \frac{1}{3c}\begin{pmatrix} u_1 & \tfrac{1}{2}u_1 & 0 & \dots & 0 & 0 \\ \tfrac{1}{2}u_1 & (u_1+u_2) & \tfrac{1}{2}u_2 & \dots & 0 & 0 \\ 0 & \tfrac{1}{2}u_2 & (u_2+u_3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (u_{n-2}+u_{n-1}) & \tfrac{1}{2}u_{n-1} \\ 0 & 0 & 0 & \dots & \tfrac{1}{2}u_{n-1} & u_{n-1} \end{pmatrix}, \tag{4·1·6}$$

where
$$u_i = \frac{1}{\tau_i} = \frac{1}{t_{i+1}-t_i}. \tag{4·1·7}$$

4·2. *The cases $n = 1$, 2 and 3*

In the special case $n = 1$ the above expansions do not apply, but the well-known result

$$W(+) = \frac{1}{2\pi}\left(\frac{-\psi_0''}{\psi_0}\right)^{\frac{1}{2}} \tag{4·2·1}$$

is easily derived as in § 2.

When $n = 2$ we have

$$(L_{n+i,\,n+j}) \sim \frac{1}{3c}\begin{pmatrix} u_1 & \tfrac{1}{2}u_1 \\ \tfrac{1}{2}u_1 & u_1 \end{pmatrix}$$

and so by inversion
$$(\mu_{ij}) \sim c\begin{pmatrix} 4\tau_1 & -2\tau_1 \\ -2\tau_1 & 4\tau_1 \end{pmatrix}$$

giving $\nu_{12} \sim -\frac{1}{2} = \cos^{-1}(\frac{2}{3}\pi)$. Thus from (2·2·2) and (2·2·3)

$$\left.\begin{aligned} W(+,+) &\sim \frac{1}{\pi^2}\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\frac{c}{(-\psi_0\psi_0'')^{\frac{1}{2}}},\\ W(+,-) &\sim \frac{1}{\pi^2}\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\frac{c}{(-\psi_0\psi_0'')^{\frac{1}{2}}}.\end{aligned}\right\} \tag{4·2·2}$$

When $n = 3$ we have $$(L_{n+i,\,n+j}) \sim \frac{1}{3c}\begin{pmatrix} u_1 & \frac{1}{2}u_1 & 0 \\ \frac{1}{2}u_1 & (u_1+u_2) & \frac{1}{2}u_2 \\ 0 & \frac{1}{2}u_2 & u_2 \end{pmatrix}$$

and so $$(\mu_{ij}) \sim \frac{c}{\tau_1+\tau_2}\begin{pmatrix} \tau_1(3\tau_1+4\tau_2) & -2\tau_1\tau_2 & \tau_1\tau_2 \\ -2\tau_1\tau_2 & 4\tau_1\tau_2 & -2\tau_1\tau_2 \\ \tau_1\tau_2 & -2\tau_1\tau_2 & (4\tau_1+3\tau_2)\,\tau_2 \end{pmatrix}.$$

Thus $$\nu_{23}, \nu_{31}, \nu_{12} = -\left(\frac{\tau_2}{3\tau_1+4\tau_2}\right)^{\frac{1}{2}}, \quad \left(\frac{\tau_1\tau_2}{(3\tau_1+4\tau_2)\,(4\tau_1+3\tau_2)}\right)^{\frac{1}{2}}, \quad -\left(\frac{\tau_1}{4\tau_1+3\tau_2}\right)^{\frac{1}{2}}.$$

Writing for short $$\left.\begin{aligned} \frac{\tau_1}{\tau_1+\tau_2} &= \frac{t_2-t_1}{t_3-t_1} = x,\\ \frac{\tau_2}{\tau_1+\tau_2} &= \frac{t_3-t_2}{t_3-t_1} = y,\end{aligned}\right\} \tag{4·2·3}$$

so that $x+y = 1$, we find from (2·2·4)

$$W(S) = \frac{1}{8\pi^3}\frac{c}{(-\psi_0\psi_0'')^{\frac{1}{2}}}\frac{1}{t_3-t_1}Q^{(S)}, \tag{4·2·4}$$

where

$$\left.\begin{aligned} Q^{(+,+,+)} &= 3-\frac{(5-2x)\,x^{\frac{1}{2}}}{(4-x)^{\frac{1}{2}}}\cos^{-1}\left(\frac{x^{\frac{1}{2}}}{2}\right)+\pi(xy)^{\frac{1}{2}}-\frac{(5-2y)\,y^{\frac{1}{2}}}{(4-y)^{\frac{1}{2}}}\cos^{-1}\left(\frac{y^{\frac{1}{2}}}{2}\right),\\ Q^{(+,-,+)} &= 3+\frac{(5-2x)\,x^{\frac{1}{2}}}{(4-x)^{\frac{1}{2}}}\cos^{-1}\left(\frac{-x^{\frac{1}{2}}}{2}\right)+\pi(xy)^{\frac{1}{2}}+\frac{(5-2y)\,y^{\frac{1}{2}}}{(4-y)^{\frac{1}{2}}}\cos^{-1}\left(\frac{-y^{\frac{1}{2}}}{2}\right),\\ Q^{(+,-,-)} &= 3-\frac{(5-2x)\,x^{\frac{1}{2}}}{(4-x)^{\frac{1}{2}}}\cos^{-1}\left(\frac{x^{\frac{1}{2}}}{2}\right)-\pi(xy)^{\frac{1}{2}}+\frac{(5-2y)\,y^{\frac{1}{2}}}{(4-y)^{\frac{1}{2}}}\cos^{-1}\left(\frac{-y^{\frac{1}{2}}}{2}\right).\end{aligned}\right\} \tag{4·2·5}$$

These three functions are plotted in figure 1. $Q^{(+,+,+)}$ and $Q^{(+,-,+)}$ are symmetrical about the mid-point $x = \frac{1}{2}$, as would be expected, whereas $Q^{(+,-,-)}$ is asymmetrical.

The probability density of a down-crossing at t_2 given up-crossings at t_1 and t_3 is proportional to $Q^{(+,-,+)}$. Figure 1 then shows that the probability density is a maximum when t_2 is mid-way between the two ends of the interval. On the other hand the curve for $Q^{(+,+,+)}$ shows that given up-crossings at both t_1 and t_3 the probability density of an *up-crossing* at an intermediate point t_2 is fairly insensitive to the position of t_2. At the mid-point, the density is actually a minimum.

4·3. *General values of n*

Exact expressions do not appear to exist in general, but upper and lower bounds for $W(S)$ may be obtained in the following way.

In equation (2·1·6) write

$$\eta_i = \frac{x_i}{(L_{ii})^{\frac{1}{2}}}, \qquad l_{ij} = \frac{L_{ij}}{(L_{ii}L_{jj})^{\frac{1}{2}}}. \tag{4·3·1}$$

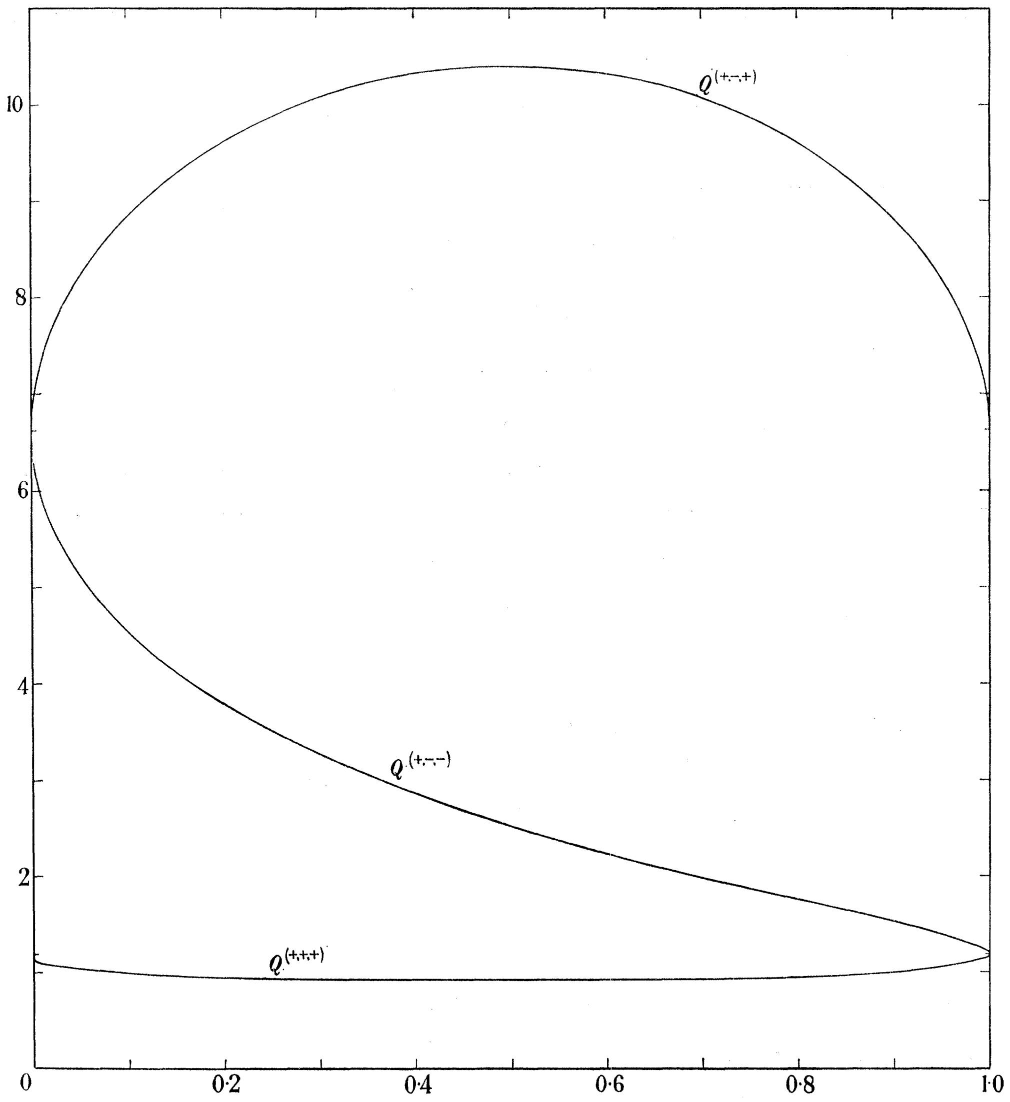

FIGURE 1. Graphs of $Q^{(+,+,+)}$, $Q^{(+,-,+)}$ and $Q^{(+,-,-)}$.

This gives
$$W(+,+,\ldots,+) = \frac{1}{(2\pi)^n\,\Delta^{\frac{1}{2}}}\,\frac{\Phi(1)}{L_{11}L_{22}\ldots L_{nn}}, \tag{4·3·2}$$

where
$$\Phi(1) = \int_0^\infty \ldots \int_0^\infty x_1 x_2 \ldots x_n \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} l_{ij}x_i x_j\right]. \tag{4·3·3}$$

Using the asymptotic formulae for Δ and L_{ij} (equations (4·1·4) and (4·1·6)) we have
$$W(+,+,\ldots,+) \sim \frac{1}{(2\pi)^n}\,\frac{3^n}{12^{\frac{1}{2}(n-1)}}\,\frac{c}{(-\psi_0\psi_0'')^{\frac{1}{2}}}\,\frac{\Phi(1)}{(\tau_1+\tau_2)(\tau_2+\tau_3)\ldots(\tau_{n-2}+\tau_{n-1})} \tag{4·3·4}$$

and
$$(l_{ij}) = \begin{pmatrix} 1 & \frac{1}{2}\left(\frac{u_1}{u_1+u_2}\right)^{\frac{1}{2}} & 0 & \ldots & 0 \\ \frac{1}{2}\left(\frac{u_1}{u_1+u_2}\right)^{\frac{1}{2}} & 1 & \frac{1}{2}\left(\frac{u_2}{u_1+u_2}\right)^{\frac{1}{2}}\left(\frac{u_2}{u_2+u_3}\right)^{\frac{1}{2}} & \ldots & 0 \\ 0 & \frac{1}{2}\left(\frac{u_2}{u_1+u_2}\right)^{\frac{1}{2}}\left(\frac{u_2}{u_2+u_3}\right)^{\frac{1}{2}} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{4·3·5}$$

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It is now easy to find bounds for Φ. For since $u_i > 0$, all the elements of (l_{ij}) adjacent to the diagonal lie between 0 and $\frac{1}{2}$. Therefore, the x_i being non-negative,

$$\sum_i x_i^2 \leqslant \sum_{i,j} l_{ij} x_i x_j \leqslant \sum_i x_i^2 + \sum x_i x_{i+1}. \tag{4·3·6}$$

In the right-hand inequality substitute

$$x_i x_j \leqslant \tfrac{1}{2}(x_i^2 + x_j^2)$$

giving

$$\sum x_i^2 \leqslant \sum_{i,j} l_{ij} x_i x_j \leqslant 2 \sum_i x_i^2.$$

Thus from (4·3·3)

$$1 \geqslant \Phi \geqslant 1/2^n. \tag{4·3·7}$$

These bounds may now be substituted in (4·3·4).

For $W(S)$, when the signs of S are not all +, one or more of the l_{ij} may be reversed in sign. Hence the left-hand inequality in (4.3.6) is not valid but may be replaced by

$$\sum_i x_i^2 - \sum_i x_i x_{i+1} \leqslant \sum_{i,j} x_i x_{i+1}.$$

Now it can be shown that

$$\lambda \sum_i x_i^2 \leqslant \sum_i x_i^2 - \sum_i x_i x_{i+1},$$

where λ is the smallest root of the equation

$$\begin{vmatrix} (1-\lambda) & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & (1-\lambda) & -\frac{1}{2} & \cdots & 0 \\ 0 & -\frac{1}{2} & (1-\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & (1-\lambda) \end{vmatrix} = 0,$$

that is to say

$$\lambda = 2\sin^2\{\pi/2(n+1)\}.$$

Since, then, in all possible cases

$$\lambda \sum_i x_i^2 \leqslant \sum_{i,j} l_{ij} x_i x_j \leqslant 2 \sum_i x_i^2$$

we have

$$\frac{1}{2^n \sin^{2n}\{\pi/2(n+1)\}} \geqslant \Phi \geqslant \frac{1}{2^n}. \tag{4·3·6}$$

Our general result then is that

$$W(S) \sim \frac{1}{(2\pi)^n} \left(\frac{3}{4}\right)^{\frac{1}{2}n} \frac{c}{(-\psi_0 \psi_0'')^{\frac{1}{2}}} \frac{2\sqrt{3}\,\Phi}{(t_3-t_1)(t_4-t_2)\ldots(t_n-t_{n-2})}, \tag{4·3·7}$$

where Φ is a function of the t_i lying between the bounds (4.3.6).

4·4. *Asymptotic behaviour of $P_m(\tau)$*

By the mean-value theorem for integrals, the integral X_n of (1·4·4) can be expressed as

$$X_n \sim \frac{1}{(2\pi)^{n-1}} \left(\frac{3}{4}\right)^{\frac{1}{2}n} \frac{c}{-\psi_0''} 2\sqrt{3}\,\Phi' K_n, \tag{4·4·1}$$

where

$$K_n = \int \ldots \int_{t_1<t_2<\ldots<t_n} \frac{dt_2 \ldots dt_{n-1}}{(t_3-t_1)(t_4-t_2)\ldots(t_n-t_{n-2})} \tag{4·4·2}$$

and Φ' is some value of Φ within the bounds (4·3·6). It can be shown that K_n is finite, and since the denominator of the integrand is homogeneous and of degree $(n-2)$, K_n is independent of (t_n-t_1), or τ. In fact when $n = 2, 3, 4, 5, \ldots$

$$K_n = 1, 1, \tfrac{1}{6}\pi^2, \tfrac{1}{3}\pi^2, \ldots. \tag{4·4·3}$$

Thus, as $\tau \to 0$, X_n tends asymptotically to a positive value independent of τ. From the expansion (1·2·11) it now appears that, for each value of m, $P_m(\tau)$ tends asymptotically to a limiting value $P_m(0)$.

This behaviour of $P_m(\tau)$ is in marked contrast to the corresponding behaviour when $\psi(t)$ is a regular function. Then, as was seen in §3·2, $P_m(\tau)$ is proportional to an increasing power of τ as m increases. A further discussion will be given in connexion with $p(n,\tau)$ (see §4·8). Meanwhile, however, we shall establish some close inequalities for $P_m(0)$ when $m = 1, 2$.

4·5. *Approximations to $P_m(0)$*

From the results of §4·2 we may evaluate $X(S)$ explicitly when $n = 2$ and 3. Thus from (4·2·1) and (4·2·2) we have

$$\left.\begin{aligned} X(+,-) &\sim \left(\frac{\sqrt{3}}{\pi}+\frac{2}{3}\right)\alpha \\ X(+,+) &\sim \left(\frac{\sqrt{3}}{\pi}-\frac{1}{3}\right)\alpha \end{aligned}\right\} \tag{4·5·1}$$

where we have written for short

$$\alpha = \frac{c}{-\psi_0''} = \frac{\psi_{0+}'''}{-6\psi_0''}. \tag{4·5·2}$$

Further, on integrating the expressions (4·2·4) with respect to t_2 over $t_1 < t_2 < t_3$ we find

$$\left.\begin{aligned} X(+,-,+) &\sim \left(\frac{3}{8\pi^2}+\frac{2}{\sqrt{3}\,\pi}-\frac{47}{288}\right)\alpha, \\ X(+,-,-) &\sim \left(\frac{3}{8\pi^2}+\frac{1}{2\sqrt{3}\,\pi}-\frac{17}{288}\right)\alpha, \\ X(+,+,+) &\sim \left(\frac{3}{8\pi^2}-\frac{1}{\sqrt{3}\,\pi}+\frac{49}{288}\right)\alpha. \end{aligned}\right\} \tag{4·5·3}$$

The identity $X(+,-,+)-X(+,+,+) = X(+,+)$ can be readily verified.

Equations (1·2·1) and (1·2·2) give, in the limit when $\tau \to 0$

$$\left.\begin{aligned} P_0 &= 1{\cdot}217\,996\alpha-(P_2+P_4+P_6+\ldots), \\ P_1 &= 0{\cdot}217\,996\alpha-(P_3+P_5+P_7+\ldots) \end{aligned}\right\} \tag{4·5·4}$$

and

$$\left.\begin{aligned} P_2 &= 0{\cdot}070\,856\alpha-(2P_4+3P_6+4P_8+\ldots), \\ P_3 &= 0{\cdot}024\,358\alpha-(2P_5+3P_7+4P_9+\ldots) \end{aligned}\right\} \tag{4·5·5}$$

whence also

$$\left.\begin{aligned} P_0 &= 1{\cdot}147\,139\alpha+(P_4+2P_6+3P_8+\ldots), \\ P_1 &= 0{\cdot}193\,638\alpha+(P_5+2P_7+3P_9+\ldots). \end{aligned}\right\} \tag{4·5·6}$$

Hence the inequalities

$$\left.\begin{aligned} 1{\cdot}147\alpha < P_0(0) < 1{\cdot}218\alpha, \\ 0{\cdot}193\alpha < P_1(0) < 0{\cdot}218\alpha, \end{aligned}\right\} \tag{4·5·7}$$

and

$$\left.\begin{aligned} 0 < P_2(0) < 0{\cdot}071\alpha, \\ 0 < P_3(0) < 0{\cdot}025\alpha. \end{aligned}\right\} \tag{4·5·8}$$

4·6. *Disproof of the 'exponential hypothesis'*

We now apply the inequalities of the previous section to a particular case which was studied experimentally by Favreau *et al.* (1956). This is the Gaussian process $f(t)$ whose spectral density is given by

$$E \propto 1/(1+\sigma^2)^2, \tag{4·6·1}$$

where σ = frequency. The covariance function ψ_τ, being the cosine transform of E, has the form

$$\psi(t) \propto (1+|t|)\,e^{-|t|} = 1-\tfrac{1}{2}t^2+\tfrac{1}{3}|t|^3-\ldots$$

and so is of the form (4·1).

The experimental results showed that the distribution of zero-crossing intervals $P_0(\tau)$ was quite close to a negative exponential. Since the mean of the distribution must be

$$\frac{1}{2W(+)} = \frac{1}{\pi}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}} = \frac{1}{\pi},$$

the only possible exponential law is

$$P_0(\tau) = (1/\pi)\,e^{-\tau/\pi}$$

which makes $P_0(0) = 1/\pi$, or, since

$$\alpha = \psi_{0+}'''/(-6\psi_0'') = \tfrac{1}{3}$$

in this case,

$$P_0(0) = 3\alpha/\pi = 0{\cdot}955\ldots\alpha. \tag{4·6·2}$$

McFadden (1956) doubted the conjecture but was unable to disprove it (McFadden's assumption that $p''(n,\tau) = 0$ when $n \geqslant 4$ is actually incorrect), since the only inequalities then available to him were the right-hand inequalities of (4·5·7). However, the left-hand inequality

$$1{\cdot}147\alpha < P_0(0)$$

is definitely contradictory to (4·6·2). Thus the exponential hypothesis is disproved.

It may be pointed out that because of certain limitations in the experiments (indicated by Favreau *et al.*), the value of $P_0(\tau)$ is liable to be underestimated at the small values of τ; so that it is not surprising that the experiments suggested a too low value of $P_0(0)$.

4·7. *Further estimates of* $P_m(0)$

If $P_4, P_5, \ldots$ are neglected in equations (4·5·5) and (4·5·6), the resulting estimates of P_0, P_1, P_2 and P_3 show that P_2/P_1 and P_3/P_2 are about equal to 1/3. Now in §1·1 it was shown that $P_m(\tau)$ tended to zero with m more rapidly than any negative power of m. It is consistent with this result to conjecture that the ratio P_{m+1}/P_m tends to a constant value. If we take roughly

$$\left.\begin{aligned} P_4(0) &\doteqdot \tfrac{1}{3}P_3(0) \doteqdot 0{\cdot}006\alpha,\\ P_5(0) &\doteqdot \tfrac{1}{3}P_4(0) \doteqdot 0{\cdot}002\alpha,\\ P_6(0) &\doteqdot \tfrac{1}{3}P_5(0) \doteqdot 0{\cdot}001\alpha,\end{aligned}\right\} \tag{4·7·1}$$

then on substituting in equations (4·5·5) and (4·5·6) we find as possibly closer approximations

$$\left.\begin{aligned} P_0(0) &\doteqdot 1{\cdot}155\alpha,\\ P_1(0) &\doteqdot 0{\cdot}196\alpha,\\ P_2(0) &\doteqdot 0{\cdot}055\alpha,\\ P_3(0) &\doteqdot 0{\cdot}017\alpha.\end{aligned}\right\} \tag{4·7·2}$$

4·8. *Asymptotic behaviour of* $p(n,\tau)$

As in § 3·3 we have

$$p(0,\tau) \sim 1, \quad p(1,\tau) \sim \frac{1}{\pi}\left(\frac{-\psi_0''}{\psi_0}\right)\tau. \tag{4·8·1}$$

When $n \geqslant 2$ we have from (1·4·1) by integration

$$p(n,\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \, 2W(+)\,[P_n(\tau'') - 2P_{n-1}(\tau'') + P_{n-2}(\tau'')]$$

provided that $p(n,0)$ and $p'(n,0)$ are both zero. This will be satisfied provided

$$p(n,\tau) = O(\tau^{1+\epsilon}), \quad \text{where} \quad \epsilon > 0.$$

Now we have seen earlier that in the singular case $P_m(\tau)$ tends to a positive value $P_m(0)$ as $\tau \to 0$. Hence by integration

$$p(n,\tau) \sim W(+)\,[P_n(0) - 2P_{n-1}(0) + P_{n-2}(0)]\,\tau^2 \tag{4·8·2}$$

as $\tau \to 0$.

From (4·5·8) and (4·5·9) we have the strict inequalities

$$\left.\begin{aligned} 0{\cdot}711\beta\tau^2 < p(2,\tau) < 0{\cdot}903\beta\tau^2, \\ 0{\cdot}051\beta\tau^2 < p(3,\tau) < 0{\cdot}243\beta\tau^2, \end{aligned}\right\} \tag{4·8·3}$$

where

$$\beta = W(+)\,\alpha = \frac{1}{12\pi}\frac{\psi_{0+}'''}{(-\psi_0\psi_0'')^{\frac{1}{2}}}. \tag{4·8·4}$$

The rough estimates (4·7·2) would yield

$$\left.\begin{aligned} p(2,\tau) \doteqdot 0{\cdot}818\beta\tau^2, \\ p(3,\tau) \doteqdot 0{\cdot}103\beta\tau^2. \end{aligned}\right\} \tag{4·8·5}$$

Equation (4·8·2) shows that, in contrast to the regular case, $p(n,\tau)$ is of order τ^2 for *all* values of n greater than or equal to 2; there is no longer a strong mutual repulsion of the zeros.

Again, a heuristic argument suggests that this result is not unreasonable. Since ψ_τ'' has no continuous derivative at the origin, the second derivative of f is, in this case, non-existent almost everywhere (cf. Bartlett 1955, chapter 5) and the first derivative $f'(t)$ may be expected to behave like a random-walk process in which the standard deviation of

$$[f'(t_1) - f'(t_2)]$$

increases like $|t_1 - t_2|^{\frac{1}{2}}$ for small time-differences. Now in the fixed interval $(0,\tau)$, if f has two or more zeros, f' has at least one. So f' is of order $\tau^{\frac{1}{2}}$ in the interval while f, by integration, is of order $\tau^{\frac{3}{2}}$. That is to say f and f' lie within intervals δf and $\delta f'$ of order $\tau^{\frac{3}{2}}$ and $\tau^{\frac{1}{2}}$, respectively. Since the joint probability density of f and f' at some fixed point t in the interval exists by hypothesis it follows that $p(n,\tau)$ is of order

$$\delta f.\delta f' = O(\tau^{\frac{3}{2}}\tau^{\frac{1}{2}}) = O(\tau^2).$$

One consequence of (4·8·2) is that, given the existence of two zeros in the interval $(0,\tau)$, the probability of $(n-2)$ further zeros in the same interval is of order $\tau^2/\tau^2 = 1$. Roughly speaking, we may say that the first two zeros serve to 'pin down' the function f and its derivative so that the probability of any further number of zeros in the interval is finite, no matter how short the interval is. However, the probability density of, say, a third zero lying somewhere between the first two depends upon the situation of the third zero relative to the first two, as was seen from the curves of figure 1.

5. A comparison of different approximations to $P_0(\tau)$

In the following we shall compare the accuracy of the approximations suggested by Rice (1945), McFadden (1956, 1958), Ehrenfeld *et al.* (1958), and Longuet-Higgins (1958; this paper is referred to as (I)), with the approximations suggested in the present paper. Discussion is purposely restricted to those methods of approximation on the basis of which numerical computation has been, or readily could be, carried out.

Two different aspects of the approximations are first considered: (*a*) their accuracy for small values of τ, both in the regular and singular case of §§ 3 and 4, and (*b*) their accuracy for large values of τ. The results are tabulated in table 1.

Then the 'narrow spectrum' approximation is considered in § 5·8, and lastly the approximations are compared numerically with experimental results obtained by analogue methods when the spectrum of $f(t)$ has certain ideal forms.

5·1. *Rice's approximation* (1945)

This has been used as a starting point for several of the later approximations. It is

$$P_0(\tau) \doteqdot \frac{W(+,-)}{W(+)} = X(+,-), \tag{5·1·1}$$

in our notation. The right-hand side, being the first term in the series (1·2·7) may also be written as $P_0^{(1)}$, where $P_0^{(N)}$ is the sum of N terms. As we have seen, the calculation of $W(+,-)$ involves the evaluation of a bivariate normal integral.

From equation (1·1·1) the error in $P_0^{(1)}$ is equal to

$$P_2+P_4+P_6+\ldots, \tag{5·1·2}$$

which is always positive. Thus $P_0^{(1)}$ always exceeds P_0.

(*a*) *Small values of* τ. In the *regular* case the highest term in the remainder is

$$P_2 \sim \tfrac{1}{280}(C_4/C_1)\,\tau^8. \tag{5·1·3}$$

Thus $P_0^{(1)}$ is correct to order τ^7 near the origin. In the *singular* case equations (4·7·1) and (4·7·2) give

$$P_2+P_4+P_6+\ldots = 0{\cdot}062\alpha, \tag{5·1·4}$$

an error of about 5%.

(*b*) *Large values of* τ. When τ is so large that $f(t)$ and $f(t+\tau)$ are uncorrelated then we have

$$P_0^{(1)} = \frac{W(+,-)}{W(+)} \sim W(-) = W(+). \tag{5·1·5}$$

P_0, on the other hand, must tend to zero, in order that $\int P_0(\tau)\,d\tau$ shall converge. Thus (5·1·5) represents also the error in $P_0^{(1)}$.

5·2. *The approximation* $P_0^{(2)}$

The approximation discussed in the present paper, namely

$$P_0^{(2)} = \frac{W(+,-)}{W(+)} - \int_{t_1<t_2<t_3} \frac{W(+,-,-)}{W(+)}\,dt_2 = X(+,-) - X(+,-,-), \tag{5·2·1}$$

appears as a natural second approximation to P_0. Its evaluation involves the single integration of $W(+,-,-)/W(+)$, which, as we have seen in § 2·2, is expressible in terms of known functions. Higher approximations

$$P_0^{(N)} = X_{2,1} - X_{3,1} + X_{4,1} - \dots (-)^{N+1} X_{N+1,1}$$

will each involve additional integrations, in general.

Equation (1·2·4) shows that the error in $P_0^{(2)}$ is equal to

$$-(P_4 + 2P_6 + 3P_8 + \dots) \tag{5·2·2}$$

which is always negative. Thus $P_0^{(2)}$ is always a lower bound for P_0.

(*a*) *Small values of* τ. In the regular case the highest term in the remainder is

$$-P_4 \sim -\frac{8}{21.(11)!}\frac{C_6}{C_1}\tau^{19}. \tag{5·2·3}$$

Thus $P_0^{(2)}$ is correct to order τ^{18} near the origin. In the singular case equations (4·7·1) give

$$-(P_4 + 2P_6 + \dots) \doteqdot -0{\cdot}008\alpha, \tag{5·2·4}$$

an error of 0·7 %.

It is clear that near the origin this approximation leaves little to be desired.

(*b*) *Large values of* τ. Asymptotically we have

$$\frac{W(+,-,-)}{W(+)} \sim W(+)^2$$

and hence

$$P_0^{(2)} \sim -W(+)^2\tau \tag{5·2·5}$$

which is $O(\tau)$ at infinity. It is clear that the approximation fails radically for large values of τ.

Indeed it will be seen generally that the approximation of P_0 by $P_0^{(N)}$ is analogous, for large τ, to the approximation of e^{-x} by a finite number of terms of its power series; the convergence of the approximation is non-uniform over $(0, \infty)$.

5·3. *The 'multiply conditioned' approximations*

In § 3·10 of his original paper (1945) Rice suggested that the approximation (5·1·1) might be improved by including in the probability density $W(+,-)$ the condition that $f(t)$ be positive at one or more given points of the range (t_1, t_2). The inclusion of just one extra point leads to a threefold normal integral that can be expressed in closed form. The inclusion of more than one point leads to fourfold and higher integrals.

The suggestion was taken up by Ehrenfeld *et al.* (1958), who refer to such approximations as 'multiply conditioned' approximations. Thus (5·1·1) is denoted by MC-0; with one condition at the mid-point of the range the approximation is MC-1, and so on. Clearly all such approximations are, like MC-0, upper bounds for the true value P_0.

(*a*) *Small values of* τ. The error in MC-1 is just equal to the probability density of a down-crossing at $t = \tau$ plus a zero in $(0, \tau)$, given that $f(0) = 0$ and $f(\tfrac{1}{2}\tau) > 0$. Now if $f(\tfrac{1}{2}\tau) > 0$, there must be at least two zeros in the interval $(0, \tfrac{1}{2}\tau)$ and/or at least two zeros in $(\tfrac{1}{2}\tau, \tau)$. If we ignore $p(5,\tau)$, $p(6,\tau)$, etc., relative to $p(4,\tau)$, the error is clearly

$$2\iint_{0<t_2<t_3<\frac{1}{2}\tau} \frac{W(+,-,+,-)}{W(+)}\,dt_2\,dt_3. \tag{5·3·1}$$

In the regular case, the neglect of $p(5,\tau)$, etc., is justified by § 3·3, and we have from (3·1·4)

$$\frac{W(+,-,+,-)}{W(+)} \sim \frac{C_4}{C_1} t_2 t_3 \tau (t_3-t_2)(\tau-t_2)(\tau-t_3).$$

Substituting in (5·3·1) we find for the error

$$\frac{39}{8!}\frac{C_4}{C_1}\tau^8. \tag{5·3·2}$$

This may be compared with (5·1·3). Clearly the order of the error is the same as in MC-0, but is less by the ratio

$$\frac{39\times 280}{8!} = \frac{13}{48}.$$

In the *singular* case the limiting value of MC-1 as $\tau \to 0$ was calculated by Rice (1958) in the case of the spectrum $(1+\sigma^2)^{-2}$ (cf. § 4·6 above), with the result

$$\text{MC-1} \sim \frac{1·251}{\pi} = 1·195\alpha,$$

since $\alpha = 1/3$ in this case. From (4·7·2) the error appears to be

$$1·195\alpha - 1·155\alpha = 0·040\alpha \tag{5·3·2}$$

or about 3 % of $P_0(0)$. Compared with MC-0, the error is reduce by about one-third.

In the case of the higher multiply-conditioned solutions, if the subintervals of $(0,\tau)$ are denoted by $(t^{(i)}, t^{(i+1)})$ (with $t^{(1)} = 0$) then the expression corresponding to (5·3·1) is

$$\sum_i \iint_{t^{(i)}<t_2<t_3<t^{(i+1)}} \frac{W(+,-,+,-)}{W(+)} \mathrm{d}t_2 \mathrm{d}t_3.$$

Since W is of order τ^6 it follows that the error is always of order τ^8.

(*b*) *Large values of* τ. When the interval τ is sufficiently large, the sign of $f(\frac{1}{2}\tau)$ becomes independent of the other conditions, and the probability of $f(\frac{1}{2}\tau)$ being positive is one-half. Hence

$$\text{MC-1} \sim \tfrac{1}{2}\text{MC-0} \sim \tfrac{1}{2}W(+). \tag{5·3·3}$$

Thus the error is reduced relative to MC-0 by one-half.

Generally, if the N 'conditioned' points in the multiply conditioned approximation MC-N are spaced so that their separation tends to infinity with τ, then

$$\text{MC-}N \sim \frac{1}{2^N}\text{MC-0} \sim \frac{1}{2^N} W(+). \tag{5·3·4}$$

5·4. *McFadden's first approximation*

McFadden (1956) gave the following approximation to $P_0(\tau)$, valid for small intervals τ

$$P_0(\tau) \doteqdot \frac{R''(\tau)}{8W(\tau)} = F(\tau),$$

say. Here $R''(\tau)$ denotes the correlation function of the 'clipped' form of $f(t)$, defined in § 1·4. From (1·4·8),

$$F(\tau) = P_0 - P_1 + P_2 - P_3 + \ldots = X(+,-) - X(+,+). \tag{5·4·1}$$

The approximation is equivalent to neglecting $P_1, P_2, \ldots$ in the above series; thus it is actually of a lower order of accuracy than $P_0^{(1)}$.

In the Gaussian case we have the well-known formula

$$R(\tau) = \frac{2}{\pi}\sin^{-1}\left(\frac{\psi(t)}{\psi_0}\right)$$

and so

$$F(\tau) = \frac{1}{2}\left(\frac{\psi_0}{-\psi_0''}\right)^{\frac{1}{2}} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\cos^{-1}\left(\frac{-\psi(t)}{\psi_0}\right). \tag{5·4·2}$$

(*a*) *Small values of* τ. In the regular case the error is of order

$$-P_1(\tau) \sim -\frac{1}{6}\frac{C_3}{C_1}\tau \tag{5·4·3}$$

and in the singular case we find by expansion in powers of τ

$$F(\tau) \sim -\psi_{0+}'''/6\psi_0'' = \alpha.$$

By comparison with (4·7·2) the error is

$$-0{\cdot}155\alpha. \tag{5·4·4}$$

(*b*) *Large values of* τ. In (5·4·1) each of the terms $X(+,-)$, $X(+,+)$ tends to 0, and so $F(\tau)$ tends to 0. The error thus vanishes.

5·5. $p_r(\tau)$ *and* $p_r^*(\tau)$

The sequence of approximations proposed in (I) depends on writing the first of equations (1·4·1) in the form

$$P_0(\tau) = -\frac{1}{W(+)}\frac{\partial^2}{\partial t_1 \partial t_n} U(t_n - t_1),$$

or

$$P_0(\tau) = \frac{1}{W(+)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} U(\tau),$$

where $U(\tau)$, $= \frac{1}{2}p(0,\tau)$, is the probability that $f(t)$ be positive throughout the interval $(0,\tau)$. Let $U(\tau)$ be replaced by the probability $U_r(t^{(1)}, \ldots, t^{(r)})$ that $f(t)$ be positive at r suitably spaced points in $(0,\tau)$. (For convenience it is supposed that the points are equally spaced and that $t^{(1)}$, $t^{(r)}$ are at the end-points.) As the number r of points is increased, U_r becomes an increasingly good approximation to U. The corresponding approximations to $P_0(\tau)$ are defined by

$$p_r(\tau) = -\frac{1}{W(+)}\frac{\partial^2}{\partial t^{(1)} \partial t^{(r)}} U_r(t^{(1)}, \ldots, t^{(r)}) \tag{5·5·1}$$

and

$$p_r^*(\tau) = \frac{1}{W(+)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} U_r(\tau). \tag{5·5·2}$$

It turns out that in the Gaussian case $p_3(\tau)$ is identical with $F(\tau)$ given by equation (5·4·2). Generally, although U_r involves an r-fold normal integral, the approximations p_r and p_r^*, which depend on the derivatives of U_r, involve only $(r-2)$-fold normal integrals. Thus p_3, p_4, p_5 and p_3^*, p_4^*, p_5^* can all be evaluated in terms of elementary functions.

(*a*) *Small values of* τ. The difference $(U_r - U)$ is equal to the probability that $f(t)$ be positive at each of the points $t^{(i)}$, *and* have a zero-crossing at some point in the interval $(0,\tau)$. Hence

$f(t)$ must have two, four or more zeros in at least one of the subintervals $(t^{(i)}, t^{(i+1)})$ (the first such zero a down-crossing) and certainly not one, three of five zeros in any of the remaining subintervals. If we neglect $p(4,\tau)$ relative to $p(2,\tau)$ and $p(3,\tau)$ the probability of such an event for the subinterval $(t^{(i)}, t^{(i+1)})$ is

$$\iint_{t^{(i)}<t_1<t_2<t^{(i+1)}} W(-,+)\,\mathrm{d}t_1\,\mathrm{d}t_2 - \iiint_{t^{(1)}<t_1<t_2;\,t^{(i)}<t_2<t_3<t^{(i+1)}} W(+,-,+)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3$$
$$-\iiint_{t^{(i)}<t_1<t_2<t^{(i+1)};\,t_2<t_3<t^{(r)}} W(-,+,-)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3. \quad (5\cdot5\cdot3)$$

Again, if $p(4,\tau)$ is neglected the probability of a pair of zeros in more than one of the subintervals is negligible, so that the events are mutually independent. So (U_r-U) is equal to the sum of $(r-1)$ expressions like (5·5·3).

On differentiating (5·5·3) partially with respect to both $t^{(1)}$ and $t^{(r)}$ the first integral vanishes identically whenever $r>2$; the other integrals also vanish except when $i=1$ or $(r-1)$. Hence we have

$$p_r(\tau)-P_0(\tau) \sim \frac{2}{W(+)}\frac{\partial^2}{\partial t^{(1)}\,\partial t^{(r)}}\iiint_{t^{(1)}<t_1<t_2;\,t^{(r-1)}<t_2<t_3<t^{(r)}} W(+,-,+)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3.$$

Substituting for $W(+,-,+)$ from (3·1·4) we find

$$p_r(\tau)-P_0(\tau) = -\tfrac{1}{3}(C_3/C_1)\,(t^{(r)}-t^{(1)})\,(t^{(r)}-t^{(r-1)})\,(t^{(r)}+2t^{(r-1)}-3t^{(1)}).$$

Now putting $(t^{(r)}-t^{(1)})=\tau$ and $(t^{(r)}-t^{(r-1)})=\tau/(r-1)$ we obtain for the error in p_r

$$-\frac{(3r-5)}{3(r-1)^3}\frac{C_3}{C_1}\tau^4. \quad (5\cdot5\cdot4)$$

Thus $p_r(\tau)$ is correct to order τ^3 (not τ^4, as was stated in (I)). In particular when $r=3$, 4, 5 the errors are, respectively,

$$-\frac{1}{6}\frac{C_3}{C_1}\tau^4, \quad -\frac{7}{81}\frac{C_3}{C_1}\tau^4, \quad -\frac{5}{96}\frac{C_3}{C_1}\tau^4. \quad (5\cdot5\cdot5)$$

The case $r=3$ is in agreement with (5·4·3).

On the other hand p_r^* involves the first term in (5·5·3) which is of a lower order. Thus

$$U_r-U=(r-1)\iint_{0<t_2<t_3<\tau/(r-1)} W(+,-)\,\mathrm{d}t_2\,\mathrm{d}t_3+O(\tau^4).$$

On substituting in (5·5·2) and using (3·2·4) we find

$$p_r^*(\tau)-P_0(\tau)=\frac{1}{r-1}P_0\left(\frac{\tau}{r-1}\right)+O(\tau^4).$$

Therefore at $\tau=0$

$$\left.\begin{aligned} p_r^* &= \left(1+\frac{1}{r-1}\right)P_0=0,\\ \frac{\mathrm{d}p_r^*}{\mathrm{d}\tau} &= \left(1+\frac{1}{(r-1)^2}\right)\frac{\mathrm{d}P_0}{\mathrm{d}\tau},\\ \frac{\mathrm{d}^2p_r^*}{\mathrm{d}\tau^2} &= \left(1+\frac{1}{(r-1)^3}\right)\frac{\mathrm{d}^2P_0}{\mathrm{d}\tau^2},\\ \frac{\mathrm{d}^3p_r^*}{\mathrm{d}\tau^3} &= \left(1+\frac{1}{(r-1)^4}\right)\frac{\mathrm{d}^3P_0}{\mathrm{d}\tau^3}.\end{aligned}\right\}$$

For example, when $r = 3$ we have

$$\frac{dp_3^*}{d\tau} = \frac{5}{4}\frac{dP_0}{d\tau},$$

a relation proved independently in (I). For $r \geqslant 3$, since $P_0(\tau) \sim (C_2/C_1)\tau$, the error in $p_r^*(\tau)$ is

$$\frac{1}{(r-1)^2}\frac{C_2}{C_1}\tau. \tag{5·5·6}$$

In the singular case we make a straightforward expansion of $p_r(\tau)$ in powers of τ. The calculations lead eventually to the following, when $r = 3, 4, 5$

$$p_3(0) = \alpha,$$

$$p_4(0) = \frac{1}{\pi}\left[\cos^{-1}\left(\frac{-7}{8}\right) + \frac{8}{3\sqrt{15}}\right]\alpha = 1·0583\alpha,$$

$$p_5(0) = \frac{1}{2\pi}\left[\left\{2\cos^{-1}\left(\frac{-11}{12}\right) - \cos^{-1}\left(\frac{7}{9}\right)\right\} + \frac{1}{16}\left\{\frac{83}{\sqrt{23}} + 11\sqrt{2}\right\}\right]\alpha$$
$$= 1·0879\alpha.$$

The corresponding expansions of $p_r^*(\tau)$ lead to

$$p_3^*(0) = 1·5\alpha,$$

$$p_4^*(0) = \frac{\alpha}{9\pi}\left[\cos^{-1}\left(\frac{-1}{4}\right) + 2\cos^{-1}\left(\frac{-3\sqrt{3}}{4\sqrt{2}}\right) + 8\cos^{-1}\left(\frac{\sqrt{3}}{2\sqrt{2}}\right) + 9\cos^{-1}\left(\frac{-7}{8}\right)\right]$$
$$= 1·3551\alpha,$$

$$p_5^*(0) = \frac{\alpha}{16\pi}\left[\left(\cos^{-1}\frac{-7}{6\sqrt{2}} + \cos^{-1}\frac{-5}{3\sqrt{3}} + \cos^{-1}\frac{-1}{4} + \cos^{-1}\frac{1}{2\sqrt{6}}\right) + 16\cos^{-1}\left(\frac{-11}{12}\right)\right.$$
$$-2\left(\cos^{-1}\frac{3\sqrt{3}}{4\sqrt{2}} + \cos^{-1}\frac{1}{3}\right) + 4\left(\cos^{-1}\frac{3}{4\sqrt{2}} - \cos^{-1}\frac{13}{8\sqrt{3}}\right)$$
$$\left. + 8\left(\cos^{-1}\frac{3}{2\sqrt{2}} - \cos^{-1}\frac{7}{9}\right) + 9\left(\cos^{-1}\frac{5}{4\sqrt{3}} + \cos^{-1}\frac{1}{3} + \cos^{-1}\frac{7}{8}\right)\right]$$
$$= 1·2899\alpha.$$

These results are plotted in figure 2 against the abscissa $1/(r-1)$, and taking $\alpha = 1$. The point labelled $P_0^{(\infty)}$ and plotted at $r = \infty$ corresponds to the estimate $P_0 \doteqdot 1·155\alpha$ of § 4·7. It will be seen that p_r and p_r^* both approach $P_0^{(\infty)}$, the one from below and the other from above.

For comparison, the other approximations to $P_0(0)$ discussed above have been plotted in the same diagram. Because they are of comparable complexity, MC-0 and MC-1 have been plotted on the same ordinates as p_4 and p_5, respectively. $P_0^{(2)}$, which involves the single integration of a known function (corresponding to a fourfold integration) is plotted level with $r = 6$. It is obviously the closest approximation.

(*b*) *Large values of* τ. As $\tau \to \infty$ so $U_r(\tau)$ tends to zero; for the probability that $f(t)$ remain of constant sign throughout the infinite interval becomes vanishingly small. Hence also p_r and p_r^* tend to zero at infinity, and the error in both p_r and p_r^* is vanishingly small.

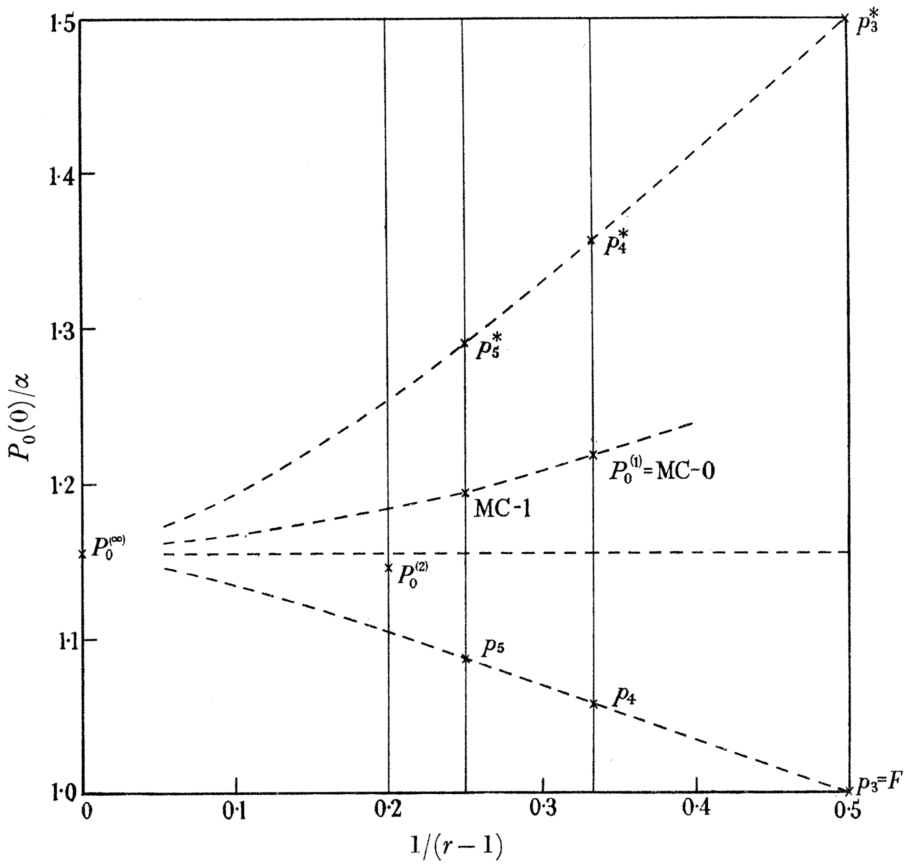

FIGURE 2. Approximations to $P_0(0)$ in the singular case.

5·6. *McFadden's second approximation*

A quite different method was suggested in a later paper by McFadden (1958). On the assumption that a given interval τ is independent of the sums of the previous $(2m+2)$ intervals $(m = 0, 1, 2, \ldots)$ McFadden derived the integral equation

$$P_0(\tau) \doteqdot X(+,-) - X(+,+) * P_0(\tau). \tag{5·6·1}$$

$X(+,-)$ and $X(+,+)$ are as in § 1·1 and a star $*$ denotes convolution:

$$F_1(\tau) * F_2(\tau) = \int_0^\tau F_1(\tau')\, F_2(\tau-\tau')\, d\tau'.$$

The solution of this (approximate) integral equation may be denoted by McF(τ).

(*a*) *Small values of* τ. In the regular case, $X(+,+) \sim P_1$ (by (1·2·1)) and so the second term in (5·6·1) is, by (3·2·6),

$$-X(+,+) * P_1(\tau) \sim -\frac{C_2 C_4}{6C_1^2} \int_0^\tau \tau'^4 (\tau-\tau')\, d\tau' = \frac{-1}{180} \frac{C_2 C_4}{C_1^2} \tau^6.$$

But we saw that $X(+,-)$, or $P_0^{(1)}$ is correct to order τ^7, so that the major part of the error is in fact due to the second term. This corresponds to the fact that for small intervals at least the assumption of independence is invalid (Palmer 1956).

In the singular case the second term is zero at the origin and so

$$\text{McF}(\tau) = X(+,-) = 1{\cdot}218\alpha.$$

The error is therefore the same as in $P_0^{(1)}$ (see equation (5·1·4)).

(*b*) *Large values of* τ. Since $X(+,-)$ and $X(+,+)$ both become equal to $W(+)$ at infinity. The solution to the limiting equation

$$F(\tau) \doteqdot W(+)-W(+)\int_0^\tau F(\tau-\tau')\,d\tau'$$

is
$$F(\tau) \doteqdot W(+)\,e^{-W(+)\tau},$$

a negative exponential. Hence we expect $McF(\tau)$ to tend to zero exponentially at infinity.

The results of §§ 5·1 and 5·6 are summarized in table 1.

TABLE 1. COMPARISON OF DIFFERENT APPROXIMATIONS TO $P_0(\tau)$

approximation	error for small τ: regular case	error for small τ: singular case	error for large τ
$P_0^{(1)} = \text{MC-0}$	$\frac{1}{280}\frac{C_4}{C_1}\tau^8$	$0{\cdot}062\alpha$	$W(+)$
$P_0^{(2)}$	$-\frac{8}{21.(11)!}\frac{C_6}{C_1}\tau^{19}$	$-0{\cdot}008\alpha$	$-W(+)^2\tau$
MC-1	$\frac{39}{8!}\frac{C_4}{C_1}\tau^8$	$0{\cdot}040\alpha$	$\frac{1}{2}W(+)$
$p_3 = F$	$-\frac{1}{6}\frac{C_3}{C_1}\tau^4$	$-0{\cdot}155\alpha$	$o(1)$
p_4	$-\frac{7}{81}\frac{C_3}{C_1}\tau^4$	$-0{\cdot}097\alpha$	$o(1)$
p_5	$-\frac{5}{96}\frac{C_3}{C_1}\tau^4$	$-0{\cdot}067\alpha$	$o(1)$
p_3^*	$\frac{1}{4}\frac{C_2}{C_1}\tau$	$0{\cdot}345\alpha$	$o(1)$
p_4^*	$\frac{1}{9}\frac{C_2}{C_1}\tau$	$0{\cdot}200\alpha$	$o(1)$
p_5^*	$\frac{1}{16}\frac{C_2}{C_1}\tau$	$0{\cdot}135\alpha$	$o(1)$
McF	$-\frac{1}{180}\frac{C_2C_4}{C_1^2}\tau^6$	$0{\cdot}062\alpha$	$o(1)$

5·7. *The narrow-band approximation*

Let $E(\sigma)$ denote the spectral density of $f(t)$, related to $\psi(t)$ by the equations

$$\left.\begin{aligned} E(\sigma) &= \frac{2}{\pi}\int_0^\infty \psi(t)\cos\sigma t\,dt,\\ \psi(t) &= \int_0^\infty E(\sigma)\cos\sigma t\,d\sigma.\end{aligned}\right\}$$

The mean frequency of the spectrum is defined as $\bar{\sigma} = m_1/m_0$, where

$$m_r = \int_0^\infty E(\sigma)\,\sigma^r\,d\sigma$$

is the rth moment of the spectrum. It is convenient to write also

$$\mu_r = \int_0^\infty E(\sigma)\,(\sigma-\bar{\sigma})^r\,d\sigma.$$

The spectrum is said to be *narrow* if

$$\frac{\mu_2}{\bar{\sigma}^2\mu_0} = \delta^2 \ll 1$$

and it can be shown (I) that in that case $\psi(t)$ has the form

$$\psi(t) = A(t)\cos\bar{\sigma}t + O(\delta\bar{\sigma}t)^3, \tag{5·7·1}$$

where $A(t)$ is a slowly varying function of t

$$A(t) = \mu_0 - \tfrac{1}{2}\mu_2 t^2 = \psi_0(1-\tfrac{1}{2}\delta^2\bar{\sigma}^2t^2).$$

Under these conditions one expects $f(t)$ to have the form of a sine wave of almost constant frequency $\bar{\sigma}$ and slowly varying amplitude, so that the greater part of the distribution $P_0(\tau)$ lies within the neighbourhood of $\tau_0 = \pi/\bar{\sigma}$. In fact the approximation $F(\tau)$ of equation (5·4·2) reduces to

$$F(\tau) \doteqdot \frac{1}{2\delta\tau_0[1+(\tau-\tau_0)^2/(\delta\tau_0)^2]^{\frac{3}{2}}} \tag{5·7·2}$$

provided $(\tau-\tau_0)$ is comparable with $\delta\tau_0$. It can be shown that all the other approximations discussed in this paper have the same limiting form as $\delta\to 0$. This approximation will be called the narrow-band approximation and will be denoted by $\mathrm{NB}(\tau)$. It has the following properties:

(1) It is symmetrical about the mean point $\tau = \tau_0$.

(2) For large values of $(\tau-\tau_0)$ it is of order $|\tau-\tau_0|^{-3}$, and so has no second moment.

(3) The maximum probability density is

$$\mathrm{NB}(\tau_0) = \frac{1}{2\delta\tau_0}.$$

(4) The width of the curve where it reaches half its maximum height is given by

$$2\times(2^{\frac{2}{3}}-1)^{\frac{1}{2}}\,\delta\tau_0 = 1{\cdot}533\delta\tau_0.$$

(5) The cumulative distribution function is

$$\int_{-\infty}^{\tau}\mathrm{NB}(\tau)\,\mathrm{d}\tau = \frac{(\tau-\tau_0)/(\delta\tau_0)}{2[1+(\tau-\tau_0)^2/(\delta\tau_0)^2]^{\frac{1}{2}}} + \tfrac{1}{2}.$$

(6) Hence the quartiles are given by

$$\frac{\tau-\tau_0}{\delta\tau_0} = \frac{1}{\sqrt{3}} = 0{\cdot}5774$$

and the interquartile range is $1{\cdot}155\delta\tau_0$.

5·8. *Numerical computations*

In this last section, the various approximations to $P_0(\tau)$ described in §§ 5·1 to 5·7 are compared by numerical computation over values of τ not necessarily very small or very large. Where possible, the results are compared with those obtained experimentally by analogue methods (Favreau *et al.* 1956; Blötekjaer 1958).

Only these approximations are shown which are the highest of their type at present available. For example, MC-0 is not shown if MC-1 is available, and p_3, p_4 are not shown if p_5 is available.

Figures 3 to 7 show the approximations p_5, p_5^*, $P_0^{(2)}$, McF and MC-0 (or MC-1) for spectra of the form

$$E(\sigma) = \sigma^{2n}/(1+\sigma^2)^m$$

(see table 2). In two cases (figures 3 and 6) the spectra are $O(\sigma^{-4})$ at infinity so that ψ has the singularity discussed in § 4. Figure 3 corresponds to the case discussed in § 4·6, where it was shown that $P_0(\tau)$ is not a negative exponential, as the observations might suggest.

The experimental results of Favreau *et al.* are indicated in figure 3 by the plotted points (the vertical lines indicate the estimated uncertainty of the observations); in figures 4 to 7 the experimental values (Favreau *et al.*) are shown by broken curves.

The curves which form lower and upper bounds of P_0—namely $P_0^{(2)}$ and MC-0 or MC-1—are drawn rather more heavily than the others. From figures 3, 6 and 7 it will be seen that at small values of τ the experimental points lie considerably below the theoretical lower bound. This implies that in some other parts of the curve the experimental points must be too high, since the total area under each curve must be unity.

Table 2

figure	$E(\sigma)$	$\psi(t)$
3	$(1+\sigma^2)^{-2}$	$e^{-\|t\|}(1+\|t\|)$
4	$(1+\sigma^2)^{-3}$	$e^{-\|t\|}(1+\|t\|+\frac{1}{3}t^2)$
5	$(1+\sigma^2)^{-5}$	$e^{-\|t\|}(1+\|t\|+\frac{3}{7}t^2+\frac{2}{21}\|t\|^3+\frac{1}{105}t^4)$
6	$\sigma^4(1+\sigma^2)^{-4}$	$e^{-\|t\|}(1+\|t\|-2t^2+\frac{1}{3}\|t\|^3)$
7	$\sigma^4(1+\sigma^2)^{-5}$	$e^{-\|t\|}(1+\|t\|-\frac{1}{3}t^2-\frac{2}{3}\|t\|^3+\frac{1}{9}t^4)$

Nevertheless, there is substantial agreement between the experimental points and the three approximations p_5^*, $P_0^{(2)}$ and McF. For all except small values of τ the agreement with p_5 is less good than with p_5^*.

Figure 8 shows a similar study in the case of the Butterworth spectrum

$$E(\sigma) = 1/(1+\sigma^{14}),$$

which has a fairly sharp cut-off at about $\sigma = 1$. The plotted points are those of Favreau *et al.* (1956). Evidently the agreement between the observations and the theoretical curves p_5^*, $P_0^{(2)}$ and McF is quite close. In the range $6 < \tau < 8$, where $P_0^{(2)}$ is the uppermost of the three curves $P_0^{(2)}$ must also be the closest approximation, since it is a lower bound.

For the low-pass spectrum

$$E(\sigma) = \begin{cases} 1 & (0 < \sigma < 1), \\ 0 & (1 < \sigma < \infty), \end{cases}$$

the best experimental results available appear to be those of Blötekjaer (1958), which are represented by the broken curve (B) in figure 9. It will be seen that the agreement with McF is fairly good, with p_5^* somewhat better, and with $P_0^{(2)}$ very close indeed, as far as $\tau = 14$.

For the band-pass spectrum

$$E(\sigma) = \begin{cases} 0 & (0 < \sigma < \sigma_1), \\ 1 & (\sigma_1 < \sigma < 1), \\ 0 & (1 < \sigma < \infty), \end{cases}$$

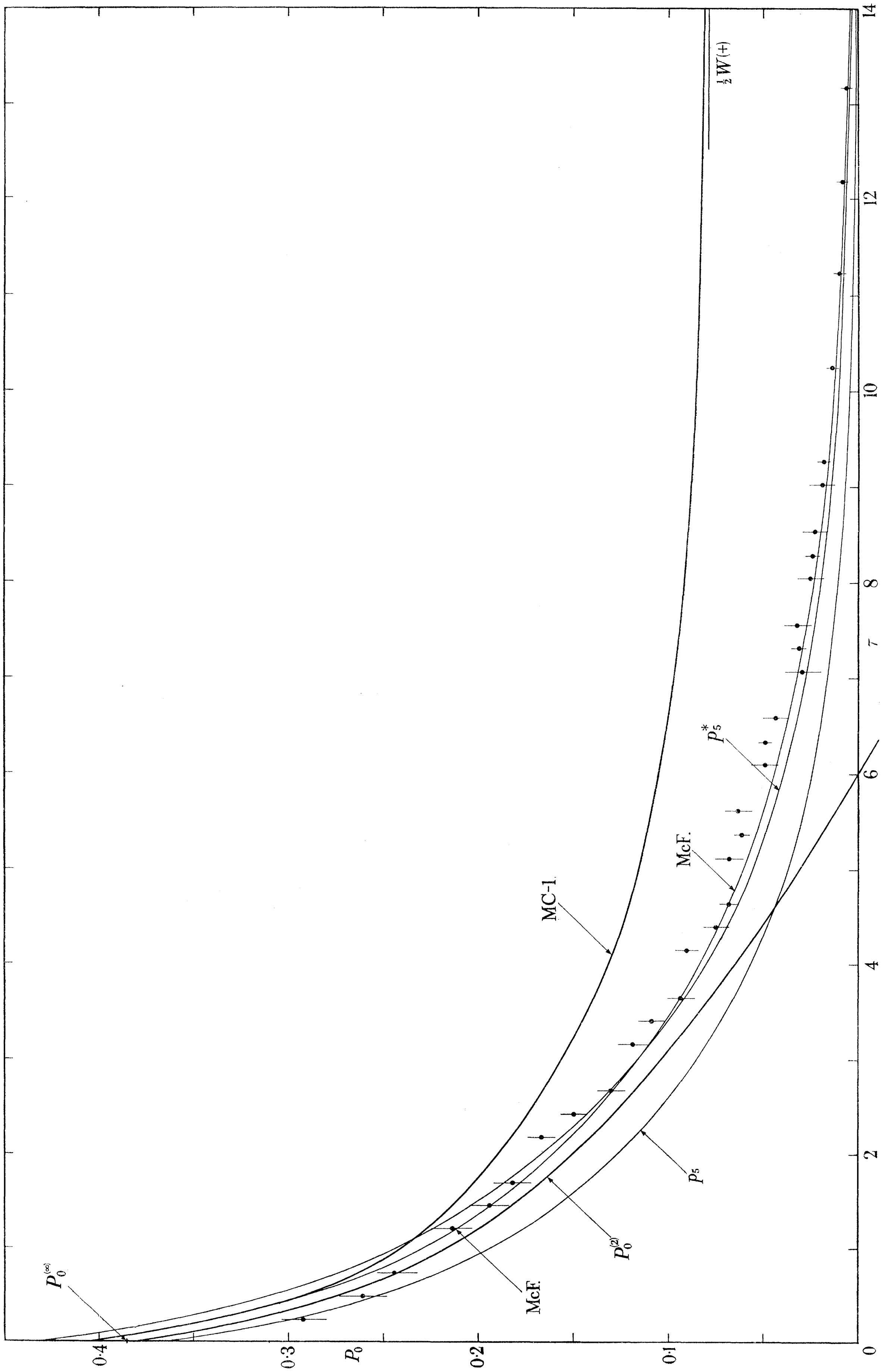

FIGURE 3. Approximations to $P_0(\tau)$ for the spectrum $E = (1+\sigma^2)^{-2}$. The plotted points are the experimental results of Favreau, Low and Pfeffer.

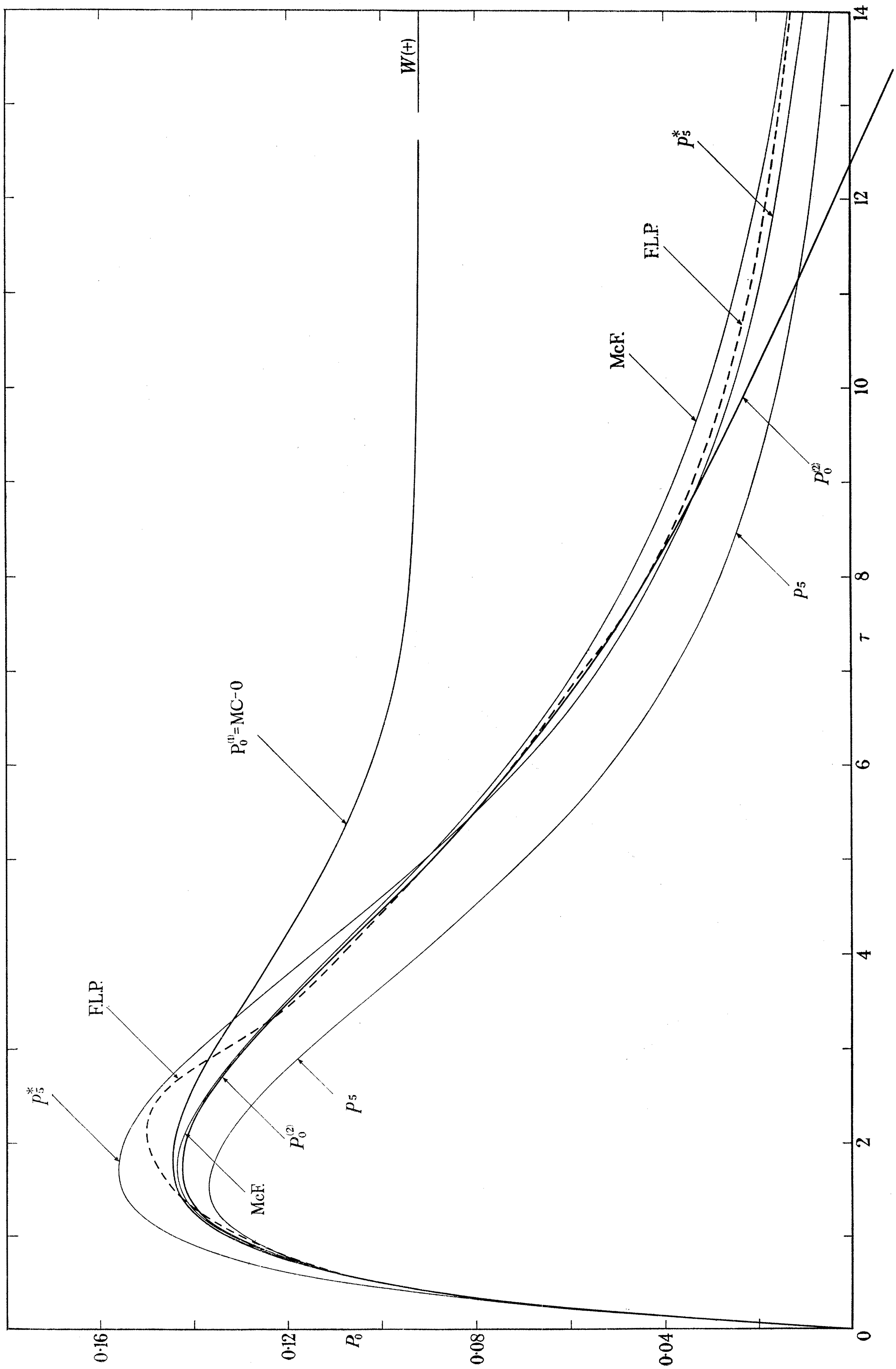

FIGURE 4. Approximations to $P_0(\tau)$ for the spectrum $E=(1+\sigma^2)^{-3}$.

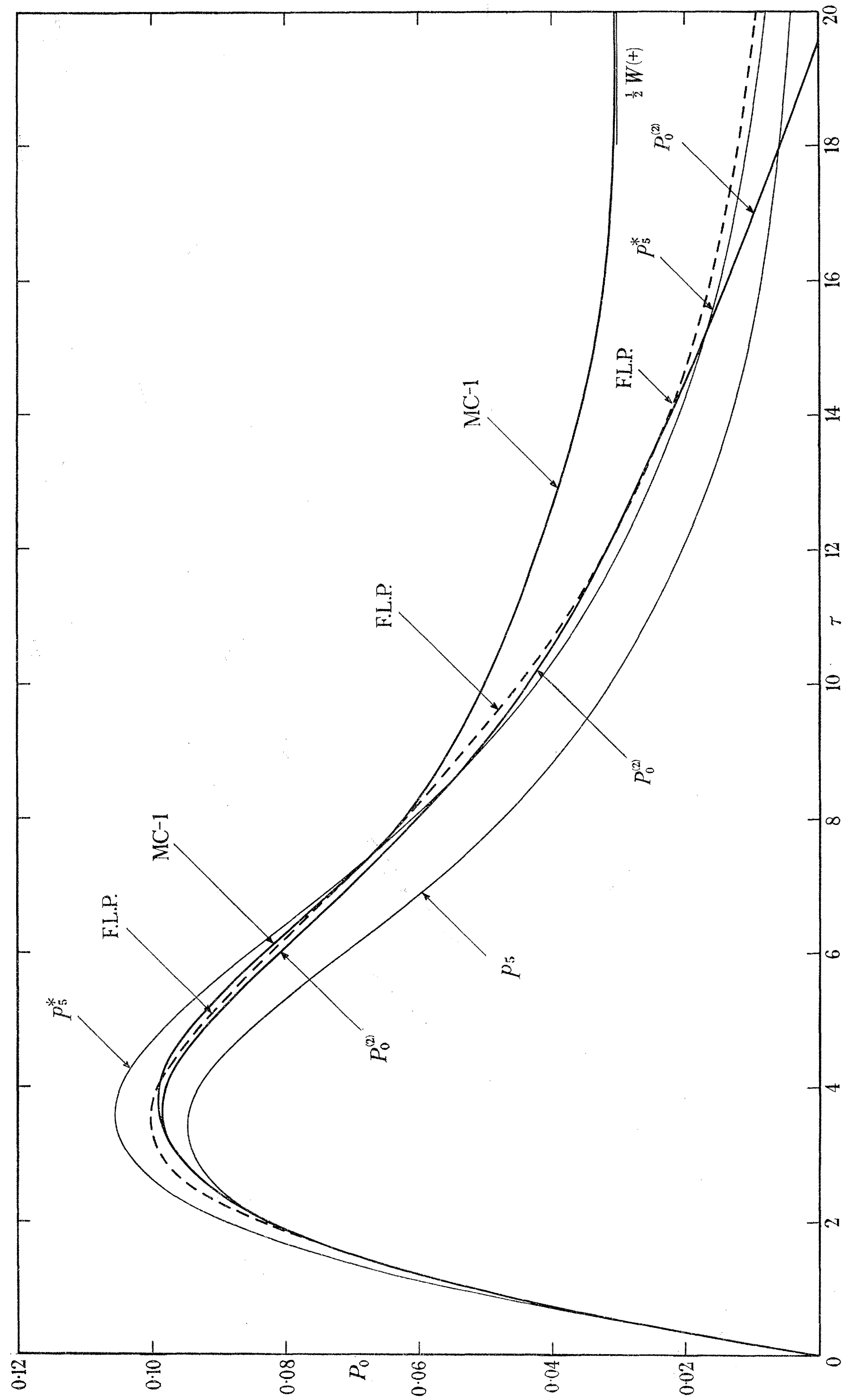

FIGURE 5. Approximations to $P_0(\tau)$ for the spectrum $E = (1+\sigma^2)^{-5}$.

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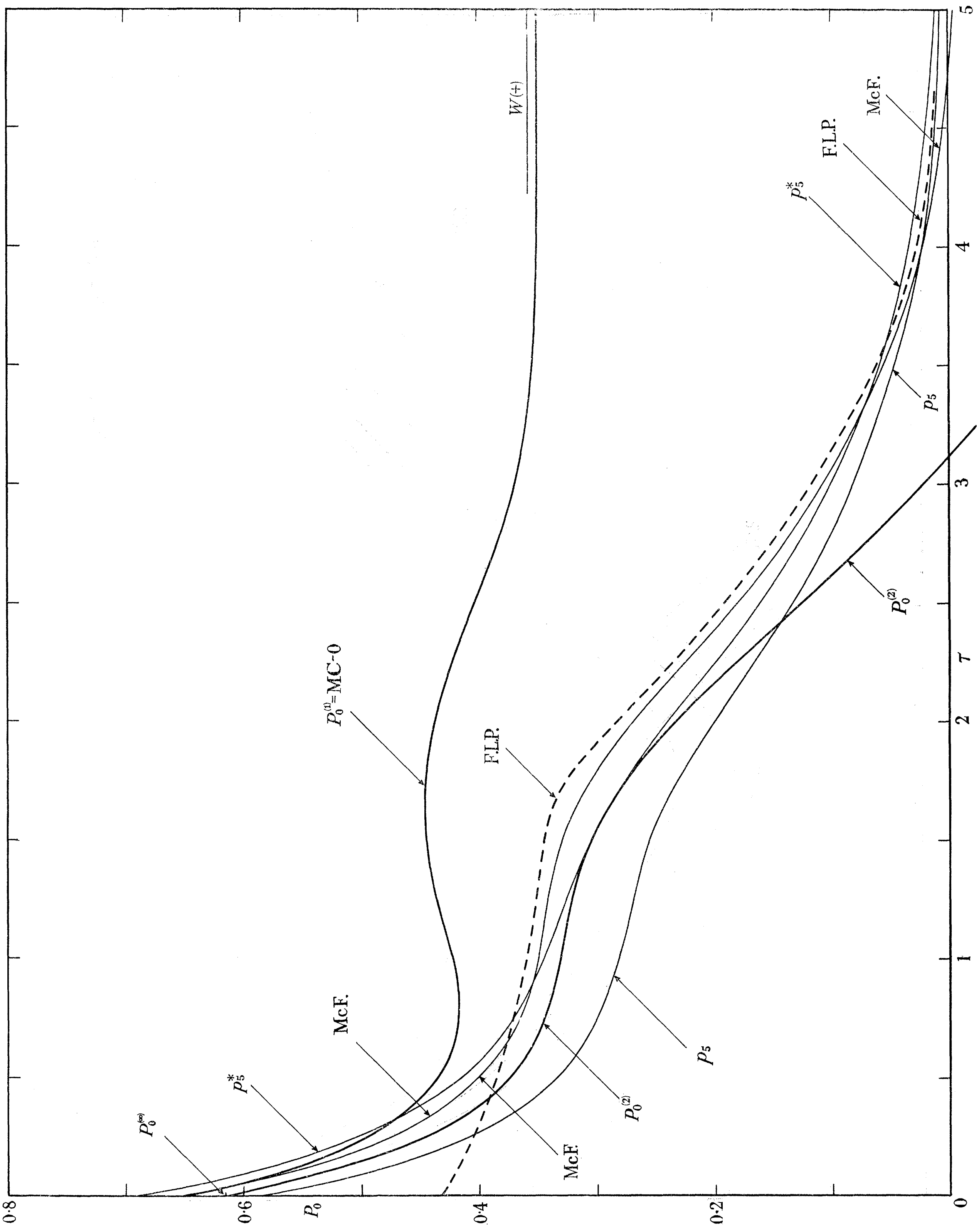

FIGURE 6. Approximations to $P_0(\tau)$ for the spectrum $E = \sigma^4(1+\sigma^2)^{-4}$.

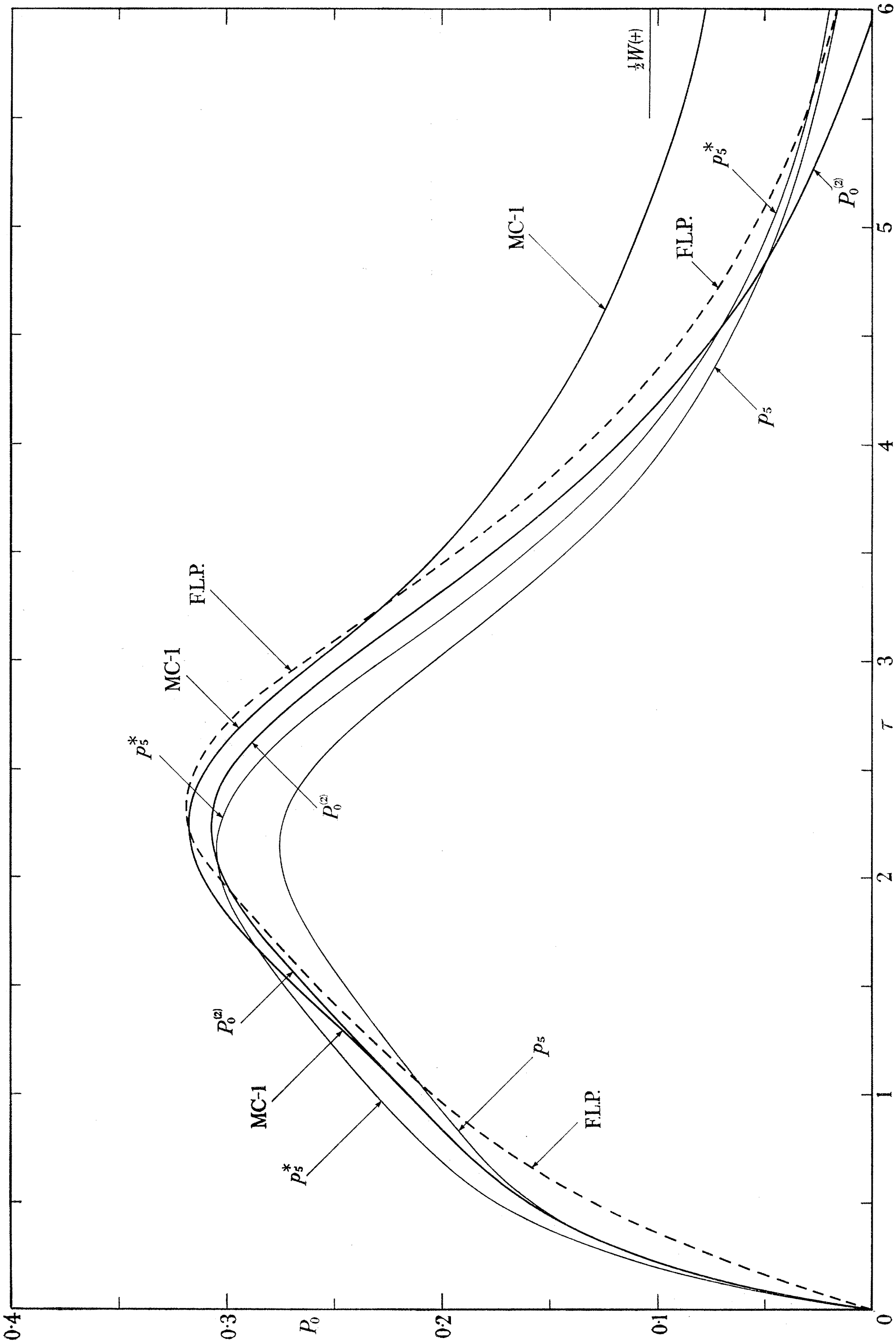

FIGURE 7. Approximations to $P_0(\tau)$ for the spectrum $E = \sigma^4(1+\sigma^2)^{-5}$.

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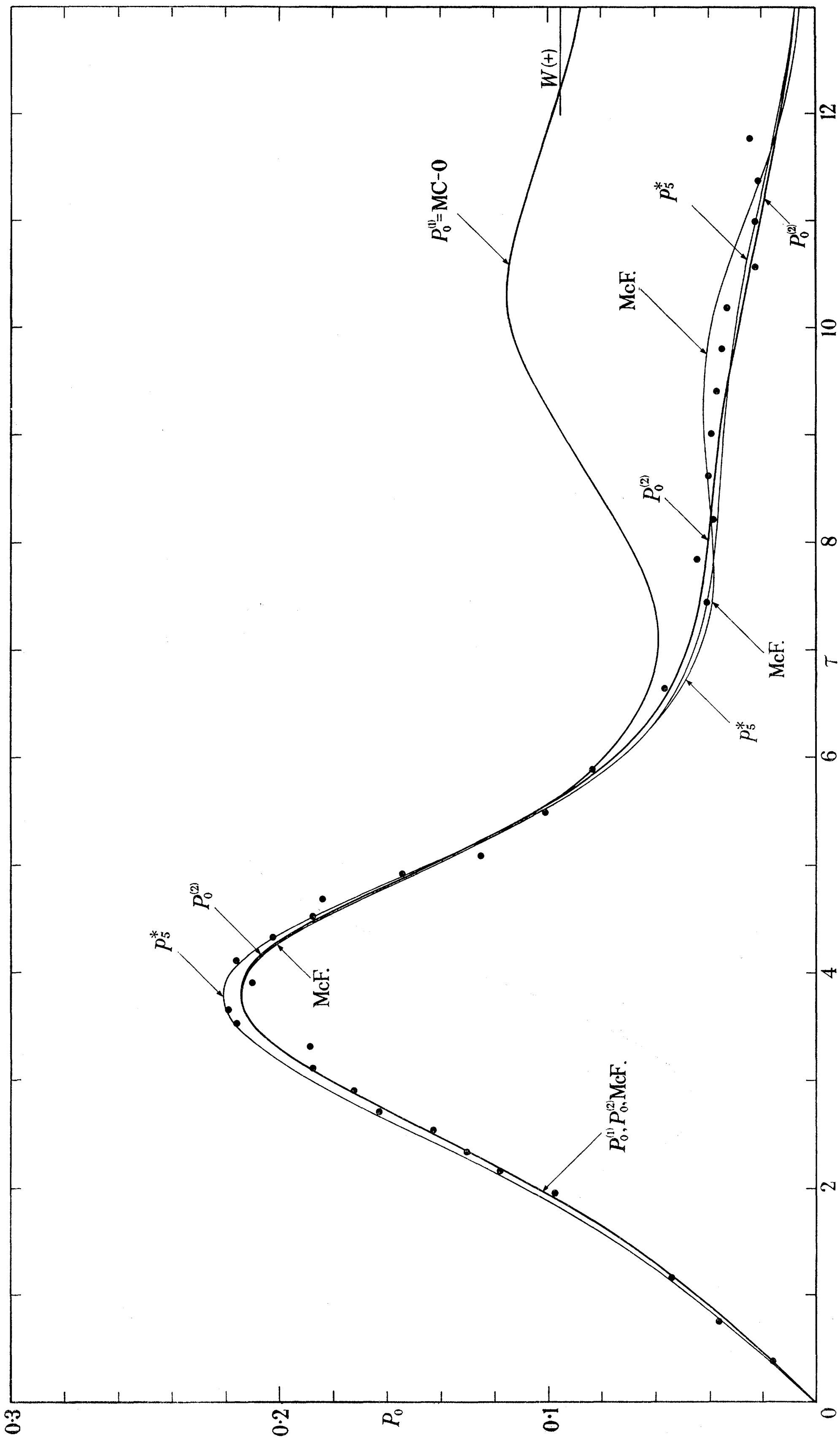

FIGURE 8. Approximations to $P_0(\tau)$ for the spectrum $E = (1+\sigma^{14})^{-1}$. The plotted points are the experimental results of Favreau, Low & Pfeffer.

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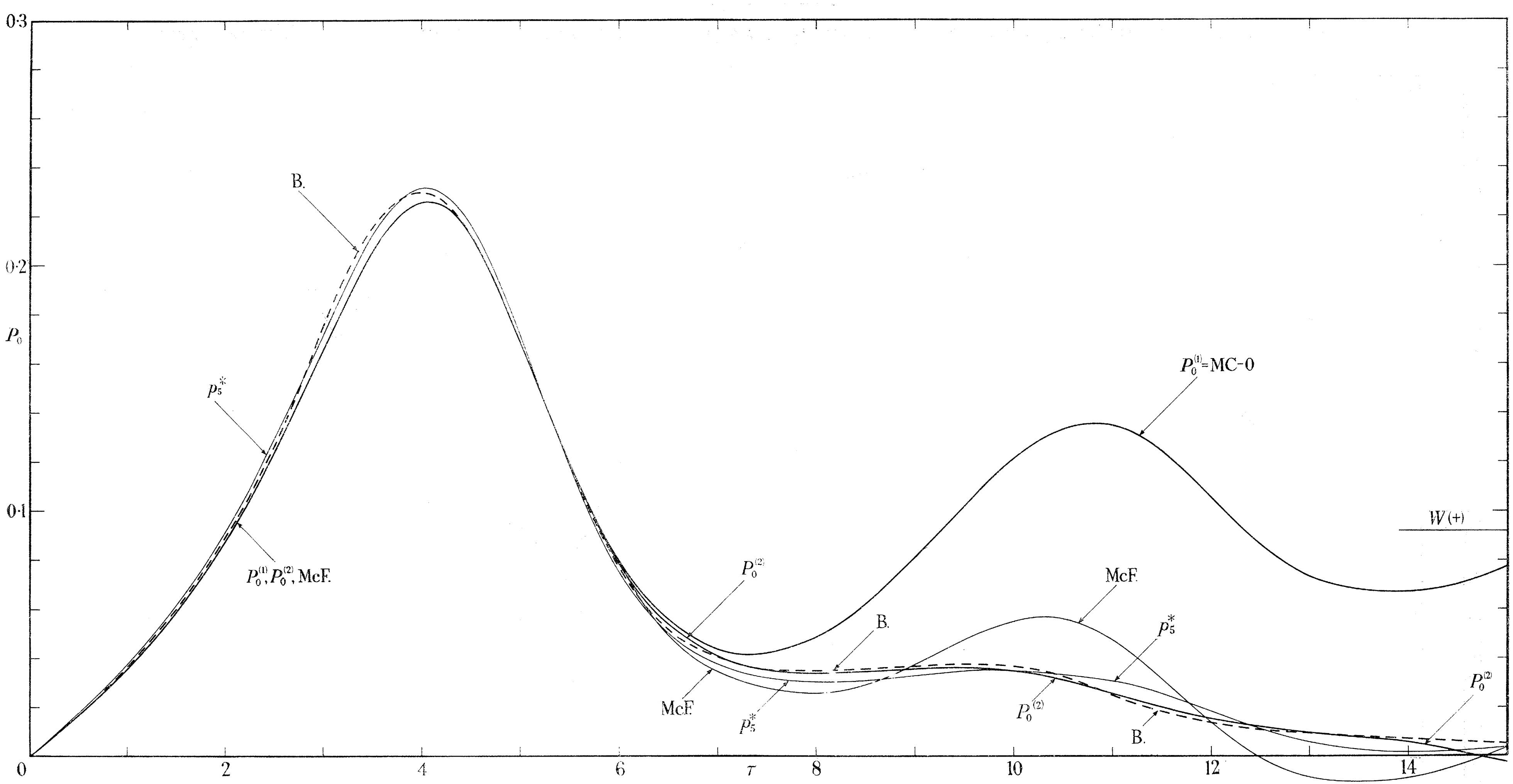

FIGURE 9. Approximations to $P_0(\tau)$ for a low-pass spectrum (cut-off at $\sigma = 1$).

$P_0^{(2)}$ can be expected to be an even closer approximation than that for the low-pass spectrum, since some of the low frequencies will have been eliminated. Accordingly $P_0^{(2)}$ has been plotted in figure 10 for $\sigma_1 = 0, \frac{1}{2}, \frac{3}{4}$ and $\frac{7}{8}$. It will be seen that as σ_1 approaches 1 and the spectrum becomes narrower so the distribution also becomes narrower and the height of

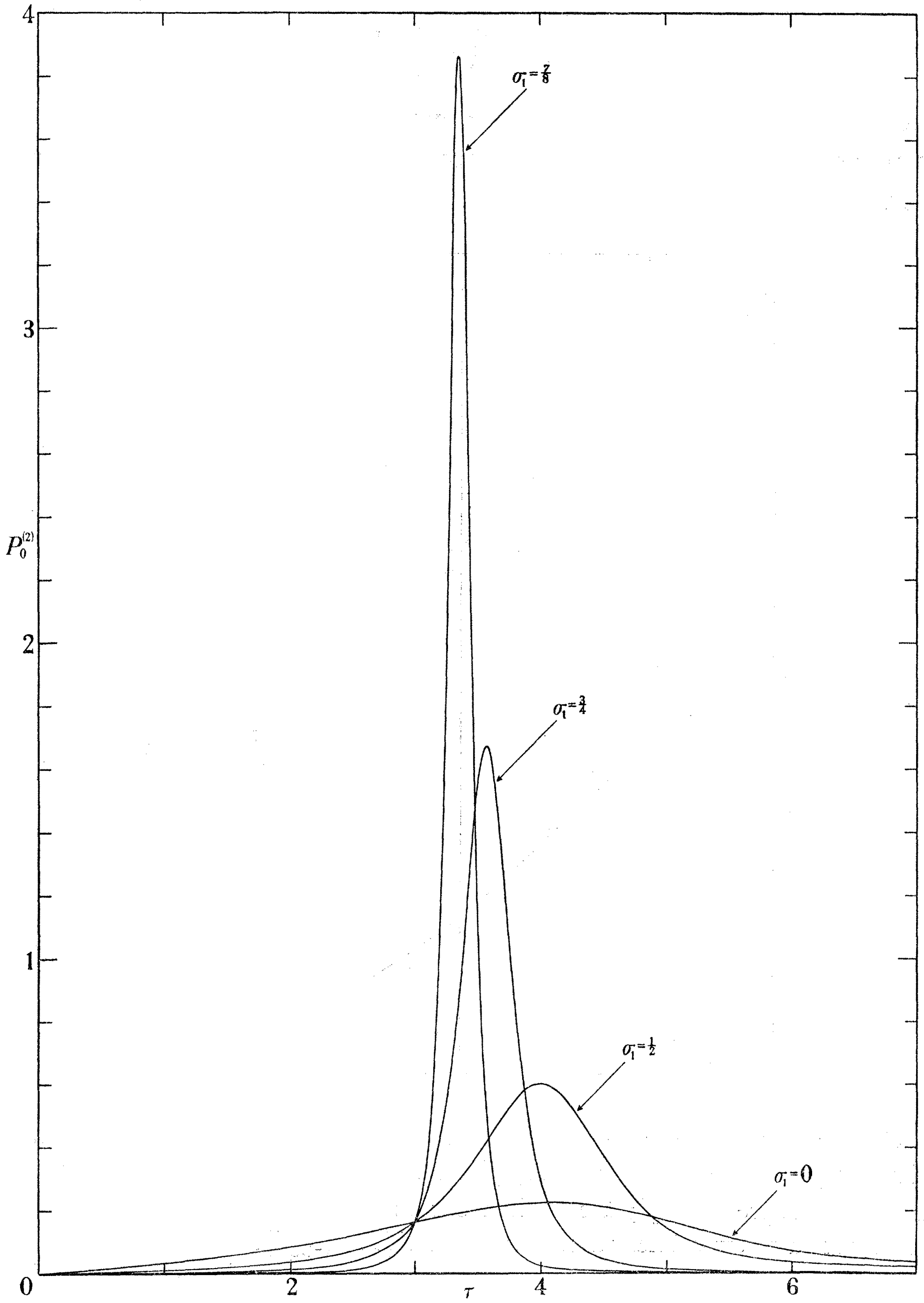

FIGURE 10. Computed values of $P_0(\tau)$ for a band-pass spectrum (low and high cut-offs at σ_1 and 1, respectively).

the maximum probability density is increased. The position and height of the maximum, and the width of the curve at half the maximum height are shown by the full curves (*a*), (*b*) and (*c*) in figure 11. (For plotting these curves the distributions $P_0^{(2)}$ for $\sigma_1 = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$ and $0{\cdot}6$ were also computed.)

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A comparison may be made with the narrow-band approximation of § 5·7, which gives for the abscissa of the maximum

$$\tau_0 = 2\pi(1+\sigma_1)^{-1}$$

and for the height and width of the distribution $(2\delta\tau_0)^{-1}$ and $1{\cdot}533\delta\tau_0$ respectively, where

$$\delta\tau_0 = \frac{2\pi}{\sqrt{3}}\frac{1-\sigma_1}{(1+\sigma_1)^2}.$$

These values are represented by the broken curves in figure 11. It will be seen that for $\sigma_1 > \frac{1}{2}$ the narrow band approximation agrees well with $P_0^{(2)}$ but there are noticeable divergences when $\sigma_1 < \frac{1}{2}$.

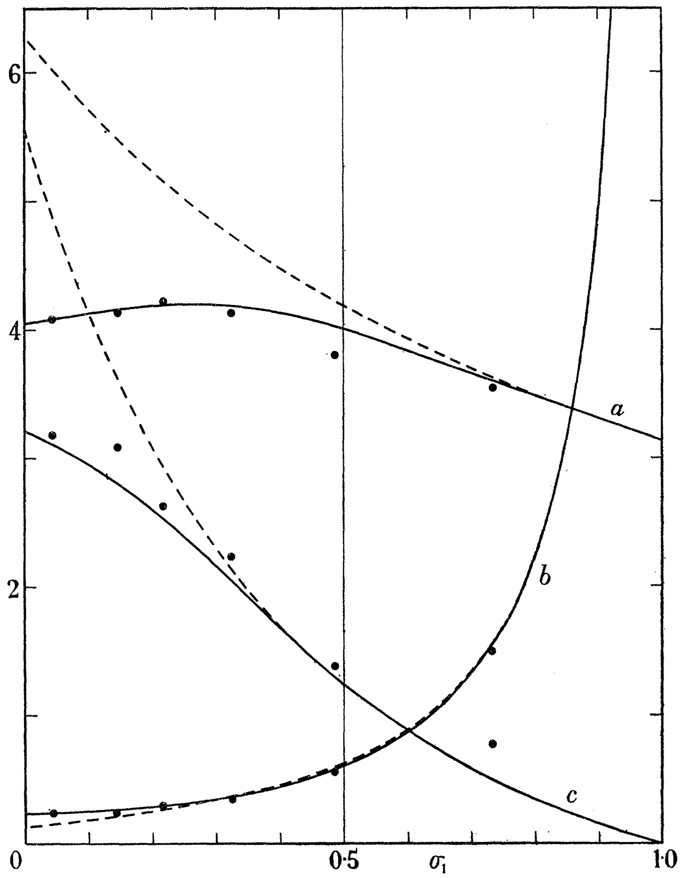

FIGURE 11. Characteristics of the distribution of intervals in a band-pass spectrum. (*a*) Abscissa of maximum; (*b*) ordinate of maximum; (*c*) width of distribution at half the maximum height. The full curves represent $P_0^{(2)}$; the broken curves the narrow-band approximation. Plotted points correspond to experimental curves of Blötekjaer (1958).

In the same figure some experimental results due to Blötekjaer (1958) have been inserted. As one would expect, they agree with $P_0^{(2)}$ rather than with NB. In particular one may note the slight negative trend in curve (*a*) as σ_1 approaches zero.

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